Goldbach's Numbers

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Abstract:Goldbach's conjectures are inseparable and both of them stem from an underlying fundamental structure of the natural numbers. Thus, one of them must consistently imply the other one, and the Goldbach's weak conjecture must imply the Goldbach's strong conjecture. Finally, all natural numbers are the Goldbach's numbers.

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Introduction

The prime numbers are fundamental multiplication blocks of the natural integers. However, there is an indication that they may be also the fundamental addition operation blocks of the natural integers. The first known statements had been made by Christian Goldbach's in 1742 known as the Goldbach's strong and the Goldbach's weak conjectures. We understand that $\mathbf{N} = \{1, 2, 3, \cdots\}$ is the set of whole or natural or n-integers. Even though the integer 1 does not satisfy the prime definition conditions it is included into primes to form the odd primes subset $\Pi = \{1, 3, 5, \cdots\},$. and this set will be our future working set of the primes.

> GOLDBACH'S STRONG CONJECTURE:
> *Every even integer is represented by a sum of two odd not necessarily distinct primes.*
>
> GOLDBACH'S WEAK CONJECTURE:
> *Every odd integer greater than 1 is represented by a sum of a three odd not necessarily distinct primes.*

Introduction of 1 in the set of the primes extends the strong conjecture to set of all even n-integers and the week conjecture to the set of all odd n-integers. Since 2 is relevant only at low even, $4 = 2 + 2 = 3 + 1$, and odd, $5 = 2 + 2 + 1 = 3 + 1 + 1$ and $7 = 2 + 2 + 3 = 5 + 1 + 1$ integers, both conjectures are reduced only to the set of the odd primes.

Definition: *The collections of all even and of all odd numbers generated by the sum of two and three primes over the set of all primes are 2Π or $2G$ and 3Π or $3G$ numbers, or shortly the Goldbach's numbers.*

The work of Harald A. Helfgott is recognized as the final proof of the Goldbach's weak conjecture. The set of all odd n-integers is created by all possible sums of three, not necessarily distinct, primes. Hence, apart from the 3Π numbers created by the prime triples there are no other odd n-integers. Thus $2\mathbf{N} + 1 = 3\Pi$. However, unless the strong Goldbach's conjecture is proven the even integers set 2Π will be only a subset of $2\mathbf{N}$ numbers. We notice that $3\mathbf{N}+1 = 3\Pi+1 = 2\mathbf{N}$ class equation holds and in addition to the $n \to n+1$ all n-integers are generated according to the following mapping diagram

$$\Pi \to 3\Pi \to 3\Pi + 1. \tag{1}$$

In spite great effort, the Goldbach's strong conjecture is not ruled out yet. The best result is the proof that almost all even integers are representable as the sum of two primes, or more precisely the probability of two primes sum representation of the large even n-integers tends to unity. The best practical result obtained by computer testing is that all even numbers less then $4 \cdot 10^{18}$ are the sum of two primes.

In this work, we will use the fact that the weak Goldbach's conjecture is proven. This implies that all odd numbers are the 3Π Goldbach's numbers. Consequently, all primes are reconstructible in a prime 3-primes representation, which further leads to the validation of the strong Goldbach's conjecture.
The first sections are devoted to the prime operations, the p-prime family constructs and the family properties of the 2G and 3G numbers. Next, the prime numbers are reconstructed over the primes, and the prime trinity equation essential to the proof of the Goldbach's strong conjecture is obtained.

Remark: Here we will specify some notation. We will use notation k, l, m, n, x, y, z, for the natural integers and p, q, r, s, t and Greek letters $\alpha, \beta, \pi, \xi, \eta, \sigma$ for the primes. For elements of 2Π and 3Π sets we will use $2a, 2b, 2c, \cdots$, and $3a, 3b, 3c, \cdots$. For the elements of the $2\mathbf{N}$ and $3\mathbf{N}$ sets we will use $2x, 2y, 2z, \cdots$, and $3x, 3y, 3z, \cdots$.

The Prime Operation Tables

Perhaps the most illustrative presentation of 2Π or 2G and 3Π or 3G numbers generated on the prime set are the prime addition tables or the prime "multiplication" matrices $\Pi \times \Pi$ and $\Pi \times \Pi \times \Pi$. The **Table 1** is a part of the multiplication matrices Π^2 and Π^3. We standardize the prime axes to (p, q, r) or (ξ, η, ζ). The matrix Π^2 is spanned by the first two prime variables. Since there is one to one correspondence between the set of the prime pairs and the prime triples, and the entries of the Π^2 and Π^3 matrices we will make the $\Pi^2 = \Pi \times \Pi$ and $\Pi^3 = \Pi \times \Pi \times \Pi$ identifications. In the global sense, the entries of the Π^2 and Π^3 are objectwise identified by the distinct elements of the 2G and 3G numbers in the sets 2Π and 3Π. Further we may write

$$2a = \xi + \eta \sim (\xi, \eta), \quad 3a = \xi + \eta + \zeta \sim (\xi, \eta, \zeta).$$

Even though the multiplication tables are redundant they preserve the algebraic operation structure and carry a great amount of useful information about the prime numbers.

Remark: We designated p and q for the row and column prime axes, and r for the third or the hight prime axes. We introduce $\mathrm{P} = \{\xi : \xi \leq p\} = \mathrm{P}_p$ for the first p primes, $\mathrm{P}_q^p = \{\xi : q \leq \xi \leq p\}$ for the prime segment $[p, \cdots, q]$ and $\Pi_q^c = \{\xi : \xi \geq p\}$ for an infinite prime collection or the prime ray, starting at a prime p. Usually, we will use p and q for the column and row axes of the matrix Π^2. However, we will use the axes $\mathcal{D}$ for the column of the pairs (p, q) and axes $\mathcal{R}$ for the row of the primes r in the matrix Π^3. The rows and the columns of the entries in Π^2 are $2\tilde{q}$ and $2\underline{p}$. We will use $3\tilde{q}$ or $A(p, q) \sim A_{pq}$ for columns and $3\underline{p}$ or R_r for the rows of the entries of the matrix Π^3.

Table 1. The prime multiplication matrices

Π^2

q	$p \to$	1	3	5	7	11	13	$*$
1		**2**	4	6	8	12	14	$*$
3		4	**6**	8	10	14	16	$*$
5		6	8	**10**	12	16	18	$*$
7		8	10	12	**4**	18	20	$*$
11		12	14	16	18	**22**	24	$*$
13		14	16	18	20	24	**26**	$*$
17		$*$	$*$	$*$	$*$	$*$	$*$	$*$
19								
23								
29								
31								
37								

Π^3

p	q	$r \to$	1	3	5	7	11	13	$*$
1	2		**3**	5	7	9	13	15	$*$
3	1		5	7	9	11	15	17	$*$
	3		7	**9**	11	13	17	19	$*$
5	1		7	9	11	13	17	19	$*$
	3		9	11	13	15	19	21	$*$
	5		11	13	**15**	17	21	23	$*$
7	1		9	11	13	15	19	21	$*$
	3		11	13	15	17	21	23	$*$
	5		13	15	17	19	23	25	$*$
	7		15	17	19	**21**	25	27	$*$
11	1		13	15	17	19	23	25	$*$
	3		15	17	19	21	25	27	$*$

Remark: Multiplication matrices are defined by their columns or rows. Calculation of the row and the columns entries of the matrix Π^2 are simple and

$$2\tilde{q} = q + \Pi, \quad 2\underline{p} = p + \Pi.$$

However, due to $3a = p+q+r = (p+q)+r = 2a_{pq} + r$, the matrix Π^3 is organized as two dimensional matrix $\mathcal{D} \times \mathcal{R}$ by row axes $\mathcal{R}$ of the primes and the column axes of the prime pairs $2a_{p,q} = p+q$ in $\mathcal{D}$. Exactly, each prime p carries a column segment $\mathrm{P}(p) = (1, 3, \cdots, p)$ of the primes, forming the proper prime pair family $p \times \mathrm{P}(p) = ((p,1), (p,3), \cdots, (p,p))$. Hence the column axes of the matrix Π^3 is direct sum of all possible proper prime families and

$$\mathcal{D} = (1,1) \dot{+} \{(3,1),(3,3)\} \dot{+} \{(5,1),(5,3),(5,5) \ \dot{+} \cdots .$$

The triple multiplication is exhibited as the multiplication of each pair by all primes from the row axes. Thus, the row A_{pq} of a pair (p,q) and a column $3\underline{r}$ in the Π^3 are

$$A_{pq} = 2a_{p,q} + \Pi, \quad 3\underline{r} = r + \mathcal{D}.$$

We notice that each column of the entries $2\underline{a}_p$ from the matrix Π^2 creates the strep

$$\mathbf{A}_p = 2\underline{a}_p + \Pi_p^{\mathrm{c}}$$

of the rows of the matrix Π^3. Clearly, there is the symmetry $(p,q) = (q,p)$ of the matrix Π^2. However, the matrix Π^3 is the triplet but not $\Pi^2 \times \Pi$ symmetric. We define the diagonals of the matrices Π^2 and Π^3 to be all $p+p$ and $p+p+p$ entries respectively, and, further, we will have in the mind mainly the upper, or over-diagonal triangular multiplication matrices.

The Prime Families

We notice that the elements $8, 10$ from the column $2\underline{a}_5$ appear only in the next, $2\underline{a}_7$, column and newer more in Π^2 matrix, and that the elements $9, 11, 13, 15$ of the column $3\underline{a}_5$ appear in the columns $3\underline{a}_7, 3\underline{a}_{11}, 3\underline{a}_{13}$, and newer more in the Π^3 matrix. The last indicates the limited reach in the multiplication matrices of the 2G and 3G numbers related to the primes p. Further, this suggests that it may be convenient to organize the primes, prime pairs prime triples and all their constructs into collections of the p-prime families.

The collections of all even, odd and all n-integers less or equal to primes p, $2p$ and $3p$ are in the natural number sets N(p), 2N(p) and 3N(p) respectively. With each p-proper prime family P $= \{1, 3, \cdots, p\}$ are associated collections K $= \{\eta : \eta < 2p\}$ and Z$= \{\zeta : \zeta < 3p\}$ of the prime domains of its 2G and 3G number families.

<u>Definition</u>: *Collections of n-integers*

$$2\mathcal{P} = \{(\xi, \eta) \sim \xi + \eta \leq 2p\}, \tag{2}$$
$$3\mathcal{P} = \{(\xi, \eta, \pi) \sim \xi + \eta + \pi \ \leq 3p\}, \tag{3}$$

are the p-prime families of the Goldbach's 2G and 3G numbers of a prime p. Subsets

$$2\mathcal{P}_o \ = \{\xi + \eta \leq 2p, \ \xi, \eta \in P\}, \tag{4}$$
$$3\mathcal{P}_o \ = \{\xi + \eta + \zeta \leq 3p, \ \xi, \eta, \zeta \in P\}, \tag{5}$$

are the p-proper families of the Goldbach's numbers. For each fixed prime p the prime sets

$$K \ = \{\xi < 2p : \exists \eta \in \Pi, \ \xi + \eta \leq 2p\},$$
$$Z \ = \{\xi \leq 3p : \exists (\eta, \zeta) \in \Pi, \ \xi + \eta + \zeta \leq 3p\}$$

are the p-proper prime extended families, and the complementary sets $\Sigma = K \backslash P$ and $\Sigma \cup \Omega = Z \backslash P$ are the proper prime family extensions.

Clearly $2\mathcal{P} \subset 2\mathrm{N}(p)$ and $3\mathcal{P} \subset 3\mathrm{N}(p)$. The following table illustrates the p-proper P and its extended K prime family as well as the proper $2\mathcal{P}_o$ and extended $2\mathcal{P}$ families of the 2G numbers for the prime $p = 11$. 2G numbers on the extension set Ω are shown in bold.

Table 2. The prime extended family $\mathcal{P}(11)$

Π	1	3	5	7	11	13	17	19	23	29	*
1	2	4	6	8	12	**14**	**18**	**20**	24	30	*
3		6	8	10	14	**16**	**20**	**22**	26	32	*
5			10	12	16	**18**	**22**	24	28	34	*
7				14	18	**20**	24	26	30	36	*
11					22	24	28	30	34	20	*
13						26	30	32	36	42	*
17							34	36	40	46	*
19								38	42	48	*
23									46	52	*

Remark: Inspecting multiplication matrices we find that the p-proper prime family of the prime $p = 5$ is $\mathrm{P}(5) = \{1, 3, 5\}$. Further, the prime $\sigma = 7$ generates $\{8, 10\}$ 2G numbers also in the $2\mathbf{N}(5)$ and must be selected into proper family extension set so that $\mathrm{K} = \{1, 3, 5, 7\}$. The primes $q = 7, 11, 13$ contribute $\{9, 11, 13, 15\}$ in the set $3\mathbf{N}(5)$, so that the p-prime proper family must be extended to the $\mathrm{Z} = \{1, 3, 5, 7, 9, 11, 13, 15\}$. We notice that the prime extensions did not introduce new members of neither $2\mathcal{P}_{\rm o}$ nor $2\mathcal{P}_{\rm o}$ family. However, this is not the general feature. For example, the member $44 \in 2N(23)$ is not originally created in the proper prime family $2\mathcal{P}_{\rm o}(23)$ but introduced by the proper prime extension member $\sigma = 31$.

Selection Function

Selection of the primes into p-prime families P, K and Z is based on the addition operation collecting 2G and 3G numbers into $2\mathcal{P}$ and $3\mathcal{P}$ sets. Therefore the prime addition operation is not only an algebraic operation but also a function selecting the primes into families and creating the Goldbach's 2G and 3G numbers.

Definition: *Mapping $\hat{\wedge} : (\xi, \eta) \to \xi + \eta \in 2N(p)$ extended to the prime triple by associativity $\hat{\wedge} : (\xi, \pi, \eta) \to \xi + \pi + \eta \in 3N(p)$ is the wedging of the primes from the prime set Π into collections $2\mathcal{P}$ and $3\mathcal{P}$ of the Goldbach's numbers, or shortly, the operation wedging the prime family.*

The function $\hat{\wedge}$ searches all pairs and triples of the primes to find those which satisfy the selection conditions, collects them in the prime family and creates the sets of the 2G and 3G numbers. However, we may make some restrictions. The primes larger then $2p$ and $3p$ do not wedge into 2G and 3G numbers. Since all primes from the prime proper set P wedge into 2G and 3G numbers one prime variable, let it be ξ, may be restricted to the proper prime set P. Further, to create all 2G and 3G numbers it is necessary to let one prime variable, let's say η, to be within K extended prime family. And finally in order to complete multiplication set for all 3G numbers only one prime, let it be ζ, should be within the Z extended prime family. Hence

$$(\xi, \eta) : \xi \leq p,\ \eta \leq \sigma_2^* < 2p,$$
$$(\xi, \eta, \zeta) : \xi \leq p,\ \eta \leq \sigma_2^* < 2p,\ \zeta \leq \sigma_3^* < 3p.$$

and all 2G and 3G number families are

$$2\mathcal{P} = \{\hat{\wedge}(\xi, \eta) :\ \xi \in \mathrm{P},\ \eta \in \mathrm{K}\}, \tag{6}$$
$$3\mathcal{P} = \{\hat{\wedge}(\xi, \eta, \zeta) :\ \xi \in \mathrm{P},\ \eta \in \mathrm{K},\ \zeta \in \mathrm{Z}\}. \tag{7}$$

Descriptive Characteristics of $1\mathcal{P}$, $2\mathcal{P}$ and $3\mathcal{P}$ Families

Here we will look at some basic properties of the $1\mathcal{P}$, $2\mathcal{P}$, and $3\mathcal{P}$ sets; for uniformity, we designated $1\mathcal{P}$ for the p-prime set. Clearly, all these sets are finite. In addition, $\mathrm{P} \subset \mathrm{K} \subset \mathrm{Z}$ and the proper prime families are ordered in the increasing order of the primes. Hence, the p-prime families are ordered by the proper set inclusion relation in the order of the primes p representatives of the family. Thus, for two consecutive primes $(p, q) : p < q$

$$1\mathcal{P}_p \subset 1\mathcal{P}_q,\ \ 2\mathcal{P}_p \subset 2\mathcal{P}_q,\ \ 3\mathcal{P}_p \subset 3\mathcal{P}_q.$$

Corollary: The extension sets of a p-proper prime family are non-empty and finite sets.

□ First, we will find the prime extended family K. By the Chebyshev-Bertrand's Postulate for all natural integers $n > 2$ there is a prime between n and $2n$. Hence, for each prime p there is an odd prime σ_1 between p and $2p$ and the set K is not empty. Further, there is a prime ξ in the proper prime family reducing $\xi + \sigma$ to a 2G number in the $2\mathcal{P}_o$ set. Hence,

$$\Sigma = \{\sigma_1, \sigma_2, \cdots, \sigma^*\},$$
$$\therefore \quad \mathrm{K} = \mathrm{P} \cup \Sigma = \{1, 3, 5, \cdots, \tau_2^*, p, \sigma_1, \sigma_2, \cdots, \sigma^*\}.$$

Since there is a prime $\sigma^* \in \Sigma$, again by the Chebyshev-Bertrand's Postulate there is a prime between σ_1 and $2\sigma_1$. Hence the extension set Ω is not empty and there is a prime ω_1 therein. Further, there is a pair (ξ, η) in the proper family reducing $\xi + \sigma + \omega$ to a 3G number in the $3\mathcal{P}_o$ set. Hence,

$$\Omega = \{\omega_1, \omega_2, \cdots, \omega^*\},$$
$$\therefore \quad \mathrm{Z} = \mathrm{K} \cup \Omega = \{1, 3, 5, \cdots, \tau_3^*, p, \sigma_1, \sigma_2, \cdots, \sigma^*, \omega_1, \omega_2, \cdots, \omega^*\}.$$

□

The wedge operation nicely collects 2G and 3G numbers into proper prime families ordered by the proper set inclusion relation. Consequently, an arbitrary 2G or 3G number is sitting in infinitely many prime families. Hence, first of all it is essential to find the most appropriate family $\mathcal{P}^2$ and $\mathcal{P}^3$ holding arbitrary Goldbach's numbers $2a$ and $3a$. For this we close the proper extension sets from below by adding the family prime p members.

Definition: *The prime p-extension closed sets are* $\overline{\Sigma} = \{p, \Sigma\}$ *in K and* $\overline{\Sigma\Omega} = \{p, \Sigma, \Omega\}$ *in Z. Corresponding p-extension closed sets* $2\overline{\Sigma}$ *and* $3\overline{\Sigma\Omega}$ *of the Goldbach's numbers are all* 2*G and* 3*G numbers in the current but not in the preceding* $\mathcal{P}^2$ *and* $\mathcal{P}^3$ *families.*

Corollary: For any 2G and 3G numbers $2a$ and $3a$ there is a prime p such that all their realizations are on the $\overline{\Sigma}$ and $\overline{\Sigma\Omega}$ prime p-extension closure sets.

□ For a 2G number $2a$ there are primes (s, t) such that $\hat{\wedge}(s, t) = 2a$. The wedge of any two primes in the p-prime proper family cannot exceed $2p$. Hence no two primes in the proper prime family P wedge to $2a$ unless $p = a$. Take the lowest $p : 2p \geq 2a$. Then $2a$ must be in the $\mathcal{P}^2(p)$ but not in its first preceding family. Since all 2G numbers in the family $\mathcal{P}^2$ are the prime domain bounded by the prime $\sigma^*(p)$ all $2a$ realizations are on the p-extension closure $\overline{\Sigma} = \{p, \Sigma\}$ and in the $2\overline{\Sigma}$ set.

For a 3G number, the primes (s, t, r) in the proper prime family cannot wedge in a $3a$ number equal to $3p$ unless $p = a$. Again take the lowest p such that $3p \geq 3a$. Then $2a$ must be in the $\mathcal{P}^2(p)$ but not in its preceding $\mathcal{P}^2(p)$ family. However, all 3G numbers therein are supported by at most the prime $\omega^*(p)$ and all $3a$ number realizations are limited to the p-extension closure $\overline{\Sigma\Omega} = \{p, \Sigma, \Omega\}$. □

The proper inclusions hold among both $2\mathrm{P} \sim \mathrm{P}^2$ and $3\mathrm{P} \sim \mathrm{P}^3$ families by their definitions. Because of the broader equivalences we made, in this part we will use the 2P and 3P sets instead, stressing particulars whenever it is necessary. We introduce the following ordered sequences of the $2\mathrm{P}_p$ and $3\mathrm{P}_p$ sets of the properly inclusive $2\mathcal{P}$ and $3\mathcal{P}$ families in the 2Π and 3Π sets

$$2\mathrm{P}_1 = 2\mathcal{P}(1),\ 2\mathrm{P}_3 = 2\mathcal{P}(3),\ 2\mathrm{P}_5 = 2\mathcal{P}(5),\ \cdots,$$
$$3\mathrm{P}_1 = 3\mathcal{P}(1),\ 3\mathrm{P}_3 = 3\mathcal{P}(3),\ 3\mathrm{P}_5 = 3\mathcal{P}(5),\ \cdots.$$

Definition: *The collections*

$$B_1 = 2P_1,\ B_3 = 2P_3 \backslash 2P_1,\ B_5 = 2P_5 \backslash 2P_3,\ \cdots,$$
$$T_1 = 3P_1,\ T_3 = 2P_3 \backslash 3P_1,\ T_5 = 3P_5 \backslash 3P_3,\ \cdots.$$

of $2G$ and $3G$ numbers are the prime sectors or cross sections of the $2G$ and $3G$ number in the 2Π and 3Π sets. The collections of all sectors are infinite sets

$$\mathcal{B} = \{B_1, B_3, B_5, B_7, \cdots\}, \quad \mathcal{T} = \{T_1, T_3, T_5, T_7, \cdots\}.$$

The following table shows 2G and 3G number sectors of the prime $p = 7$ clearly defined on the $\overline{\Sigma}$ and $\overline{\Sigma\Omega}$ prime p-extension sets.

Table 3. The prime sectors B(7) and T (7)

$\mathcal{B}^2$								$\mathcal{T}^3$										
q	$p = \cdots$	5	7	11	13	17	$*$	p	q	$r = \cdots$	5	7	11	13	17	19	23	$*$
1		$*$	$*$	12	14	$*$		1	1		$*$	$*$	$*$	$*$	19	21	$*$	
3		$*$	$*$	14	$*$	$*$		3	1		$*$	$*$	$*$	17	21	$*$	$*$	
5		$*$	12	$*$	$*$	$*$			3		$*$	$*$	17	19	$*$	$*$	$*$	
7			14	$*$	$*$	$*$		5	1		$*$	$*$	17	19	$*$	$*$	$*$	
11				$*$	$*$	$*$			3		$*$	$*$	19	21	$*$	$*$	$*$	
13					$*$	$*$			5			17	21	$*$	$*$	$*$	$*$	
17								7	1			$*$	19	19	$*$	$*$	$*$	
19									3			17	21	$*$	$*$	$*$	$*$	
23									5			19	$*$	$*$	$*$	$*$	$*$	
29									7			21	$*$	$*$	$*$	$*$	$*$	
$*$									$*$				$*$	$*$	$*$	$*$	$*$	

Corollary: The prime sectors or cross sections are finite disjoined sets ordered by the ordering of the prime numbers. If p and q, $p < q$, are two consecutive primes, all elements of the sector at p are before the elements of the sector at q, or two sectors do not have common elements unless they are the same.

□ The families 2Π and 3Π are upper bounded by $2p$ or $3p$ conditions on their element on the prime domain sets bounded by the σ^* and ω^* upper prime cats. Thus, the sectors are finite sets. The families are ordered in the increasing prime order by the set proper inclusion relation; and thus, by the sector definition its members are not in either the preceding or the next p-family. Hence, the sectors are ordered by the order of their p-primes and do not have common elements unless they are the same set. □

Equivalence Classes

Here we will take a closer look at the structure of the prime sectors. We already made an equivalence between 2G and 3G numbers and their pairs and triples or multiplets. While the numbers $2a = \hat{\Lambda}(\xi, \pi) \in 2\mathcal{P}$ and $3a = \hat{\Lambda}(\xi, \eta, \pi) \in 3\mathcal{P}$ are unique their multiplet representations are not. For example, the numbers 44 and 43 are made by all following pairs and triples

$$32 = (1, 31) = (3, 29) = (13, 19),$$
$$31 = (1, 1, 29) = (5, 3, 23) = (7, 1, 23) = (7, 5, 19) = (7, 7, 17) = (11, 1, 19) = (11, 3, 17) = (11, 7, 13).$$

Apart from these, there is no other pairs and triples in the multiplication matrices representing 32 and 31 numbers. We notice also that there is neither 2G or 3G numbers in the prime sector represented by the same prime pairs or triples. This is true in general. Further, we associate with any 2G and 3G numbers the following sets

$$2\mathrm{A} = \{(p, q) : \hat{\wedge}(p, q) = 2a\}, \quad 3\mathrm{A} = \{(p, q, r) : \hat{\wedge}(p, q, r) = 3a\}. \tag{8}$$

Two 2A or 3B sets do not have common elements unless they are defined by the same $2a$ or $3b$ numbers. Hence all 2A, as well as all 3A sets are mutually disjointed.

<u>**Definition:**</u> *Two pairs of, either doublets (p, q) and (ξ, η) or triples (p, q, r) and (ξ, η, ζ), are mutually equivalent if they are the members of the same 2A or 3A sets, or equivalently if they generate the same 2G or 3G numbers.*

The equivalence relation implies the preservation of the 2G and 3G numbers on the set of the pairs and triples. Since apart from the those in the sets 2A and 3A there is no other pairs and triples representing given 2G and 3G numbers, the sets $\overline{2\mathrm{A}}$ and $\overline{3\mathrm{A}}$ are the classes represented by the 2A and 3A numbers.

Particular 2G or 3G numbers appear in a single p-prime sector of the multiplication matrices so that each class belongs to a unique sector. Since sectors are finite the classes are finite. Each sector is a union of its classes so that the sets

$$\begin{aligned}\Pi^2 &= \bigcup \{2a\} = \bigcup_{\overline{2A}} \bigcup_{2a \in \overline{2A}} \{2a\} = \bigcup \overline{2A},\\ \Pi^3 &= \bigcup \{3a\} = \bigcup_{\overline{3A}} \bigcup_{3a \in \overline{3A}} \{3a\} = \bigcup \overline{3A}.\end{aligned}$$

are unions of the equivalence classes. For instance all equivalence classes in B(7) and T (7) are

$$\begin{aligned}&\overline{2\mathrm{A}} = \{(\xi, \eta) : \hat{\wedge}(\xi, \eta) = 2a\}, \qquad 2a \in \{12, 14\},\\ &\overline{3\mathrm{A}} = \{(\xi, \eta, \zeta) : \hat{\wedge}(\xi, \eta, \zeta) = 3a\}, \ 3a \in \{17, 19, 21\}.\\ \therefore \quad &\mathrm{B}(7) = \{\overline{12}, \overline{14}\} = \overline{12} \cup \overline{14}, \quad \mathrm{T}(7) = \{\overline{17}, \overline{19}, \overline{21}\} = \overline{17} \cup \overline{19} \cup \overline{21}.\end{aligned}$$

The Lift and the Row Operators

Characterization of the multiplication matrices Π^2 and Π^3 in the terms of the row is simple

$$a_\xi(\eta) = \hat{\wedge}(\xi, \eta) = \xi + \eta : \eta \in \Pi, \quad a_{\xi\eta}(\zeta) = \hat{\wedge}(\xi, \eta) + \zeta : \zeta \in \Pi$$

We understand the operation as a transformation lifting the prime ξ on the levels of the primes η and the pair (ξ, η) on the levels of the primes ζ by the lift operators defined on the set of all primes. We state this in the set equation form as

$$\hat{\Lambda}(\xi) = \{\hat{\eta}(xi) = \xi + \eta : \eta \in \Pi\} = \xi + \Pi, \tag{9}$$

$$\hat{\Lambda}(\xi, \eta) = \{\hat{\zeta}(\xi, \eta) = \zeta + \hat{\wedge}(\xi, \eta) : \zeta \in \Pi\} = \hat{\wedge}(\xi, \eta) + \Pi. \tag{10}$$

Related to the lift operator are the row operators $\hat{\Phi}_\xi$ and $\hat{\Phi}_{\xi\eta}$ creating the rows of the matrices Π^2 and Π^3 on all primes

$$\hat{\Phi}_\xi(\Pi) = \{\hat{\Phi}_\xi(\eta) = \xi + \eta : \eta \in \Pi\} = \xi + \Pi, \tag{11}$$

$$\hat{\Phi}_{\xi,\eta}(\Pi) = \{\hat{\Phi}_{\xi\eta}(\zeta) = \zeta + \hat{\wedge}(\xi, \eta) : \zeta \in \Pi\} = \hat{\wedge}(\xi, \eta) + \Pi. \tag{12}$$

For example

$$\hat{\Lambda}(7) \quad = \{14, 18, 20, 24, \cdots, 7+\eta, \cdots\} = \hat{\Phi}_7(\Pi), \qquad \hat{\lambda}_{11}(7) = 18 \ = \hat{\Phi}_7(11),$$
$$\hat{\Lambda}(3,5) = \{9, 11, 13, 19, \cdots, 11+\zeta, \cdots\} = \hat{\Phi}_{3,5}(\Pi), \quad \hat{\lambda}_7(3,5) = 15 = \hat{\Phi}_{3,5}(7).$$

Operations on the Prime Set

The wedge operation is not only the algebraic addition operation on the primes but also the selection function collecting primes in the prime family constructs. In the purely algebraic treatment it would be necessary to introduce some algebraic structure on the prime number set, which would require the introduction of the opposite primes. However, the wedge operation is a function and, instead, we will introduce its inverse or ant-wedge operation. The sets constructed by the ant-wedge function carry a great deal of the essential information about the prime numbers.

Definition: *Function $\hat{\vee} : (p, \xi) \to p - \xi \in 2\mathbf{N}$, $\xi \le p$ is the prime anti-wedge operation. The collection $2G^* = \{2d = \hat{\vee}(\xi, \eta) \le p-1\}$ is the p-prime anti-wedge set or the set of the d-numbers. The proper 2d-prime family is $2G^*_o = \{p - \xi : \xi \le p\} = \{0, p - \sigma^*, \cdots, p-3, p-1\}$. The collection of all such families is the prime anti-wedge set $2\Pi_*$.*

Hence, two operations on the primes are the mappings $\hat{\wedge} : (\Pi, \Pi) \to 2\Pi$ and $\hat{\vee} : (\Pi, \Pi) \to 2\Pi_*$ in the set $2\mathbf{N}$ of the even natural numbers. The number sets 2G and 2G* are equivalent to the sets of all prime pairs equipped by two operations. Exactly $2a \sim \{\xi, \eta; \hat{\wedge}\} \sim (\xi, \eta) \in \Pi \times \Pi$ and $2d \sim \{\xi, \eta; \hat{\vee}\} \sim [\xi, \eta] \in \Pi_* \times \Pi_*$. Further, the anti-wedge operation matrix Π^2_* is identified with the direct set product $\mathcal{D}^* = \Pi_* \times \Pi_*$ of all $[\xi, \eta]$ prime pairs equipped with the anti-wedge operation. In a broader context, we identify $2\Pi_*$ with $\Pi_* \times \Pi_*$ and Π^2_* sets.

Table 4. The prime difference matrix Π_*

							Π_*							
ξ	1	3	5	7	11	13	17	19	23	**29**	31	37	41	$*$
$\eta = 1$	0	2	4	6	10	12	16	18	22	28	30	36	40	$*$
3		0	2	4	8	**10**	14	16	20	26	28	34	38	$*$
5			0	2	6	**8**	12	14	18	24	26	32	36	$*$
7				0	2	**6**	**10**	12	16	22	24	30	34	$*$
11					0	**4**	**6**	**8**	12	18	20	26	30	$*$
13						**0**	**4**	**6**	**10**	16	18	24	28	$*$
17							**0**	**2**	**6**	12	14	22	24	$*$
19								**0**	**4**	**10**	12	18	22	$*$
23									**0**	**6**	**8**	14	18	$*$
29										**0**	**2**	**8**	12	$*$

The table **Table 4** is the part of the anti-wedge operation upper threangular matrix Π^2_*. The columns and the rows in the matrix are $2\underline{d}_p = p - \mathrm{P}$ and $2\tilde{d}_q = \Pi_q - q$. Each row starts with zero and each column finishes with zero. All main diagonal elements are zero and next to the main diagonal are the differences of two consecutive primes. While the d-proper prime families are clearly distinguished as 2d-columns of all primes smaller and equal to the prime p, the extended 2d-families located in the upper neighbourhood of the main diagonal are probably infinite sets, (see the bold numbers in the above table for the prime $p = 7$ extended family). For, the prime difference $2d = 2$ (on the diagonal next to the main), corresponds to the twin primes, and by the twin primes conjecture there are infinitely many such primes.

Duality, Equivalence and Conjugation

The sets 2Π and $2\Pi_*$ are created on the set (Π, Π) of the prime pairs by the wedge and anti-wedge operations. Simple inspection shows that there is a set $2\mathbf{S}$ of even integers common to both 2Π and $2\Pi_*$ numbers. We define that set to be $2\mathbf{S} = 2\Pi \cap 2\Pi_* \subset 2\mathbf{N}$. Thus, the subsets of the 2G and 2G* numbers naturally coexistent on the set $2\mathbf{S}$. Hence the natural question is if there is a relation between the sets of 2G and 2G* numbers. First we will reinstate the equivalence definition to include prime pairs on the both $\Pi \times \Pi$ and $\Pi_* \times \Pi_*$ sets, and then we will introduce the concepts of the dual pairs and conjugated integers and pairs.

Definition: *1. A prime pair $[\phi, \zeta]$ is dual of a prim pair (ξ, η) if the numbers $2d = \hat{\vee}(\phi, \zeta)$ and $2a = \hat{\wedge}(\xi, \eta)$ are identical. We write $2d \sim [\phi, \zeta] = (\xi, \eta)^* \sim 2a^*$* and use $\mathbf{D}^*$ for the dual set.

2. Two prime pairs (p, q) and (ξ, η) are equivalent in the 2Π set if they present the same $2G$ number $2a : \hat{\wedge}(\xi, \eta) = 2a = \hat{\wedge}(p, q)$. The pairs are equivalent in the $2\Pi_*$ set if they present the same $2G^*$ number $2d : \hat{\vee}(\xi, \eta) = 2d = \hat{\vee}(p, q)$. Equivalent pairs form the equivalence classes $\overline{2A}$ and $\overline{2D}$ represented by some 2A and 2D numbers.

3. Two natural numbers m and n are conjugated, $m = \tilde{n}$, if there is a prime pair (ξ, η) such that $\xi = n + \tilde{n}$ and $\eta = n - \tilde{n}$.

Remark: **1**. There is one to one correspondence between the sets $\hat{\wedge}(\Pi, \Pi)$ and $\hat{\vee}(\Pi, \Pi)$ and the multiplication matrix sets Π^2 and Π^2_*. Further, these sets are in the one to one correspondence with the collection of all prime pairs $\Pi \times \Pi$, and

$$|\hat{\wedge}(\Pi, \Pi)| = |\hat{\vee}(\Pi, \Pi)| = |\Pi^2| = |\Pi \times \Pi|.$$

However, the sets 2Π and $2\Pi_*$ are the sets of all distinct 2A and 2D representatives of the equivalent pairs, so that $|2\Pi| < |\Pi \times \Pi|$ and $|2\Pi_*| < |\Pi \times \Pi|$

2. The pair equivalence in the Π^2 set does not translate into par equivalence in the Π^2_* set unless $\overline{2\mathrm{A}}$ and $\overline{2\mathrm{D}}$ are each a single element classes. For, if $(p, q) \sim (\xi, \eta)$ and $[p, q] \sim [\xi, \eta]$ then

$$\hat{\wedge}(p, q) = \hat{\wedge}(\xi, \eta), \quad \hat{\vee}[p, q] = \hat{\vee}[\xi, \eta] \quad \therefore \quad \xi = p, \quad \eta = q.$$

3. Conjugation definition implies $2n = \hat{\wedge}(\xi, \eta)$ and $2\tilde{n} = \hat{\vee}(\xi, \eta)$. Hence the twice of the conjugated numbers are exactly 2G and 2G* numbers. Consequently the sets Π^2 and Π^2_* are mutually conjugated.

4. Since the set $2\mathbf{S}$ is an intersection of the set of all pairs and their conjugated pairs, clearly, all dual numbers $2a^* = 2d$ are in the $2\mathbf{S}$. Hence $2\mathbf{S}$ the set is the set $\mathbf{D}^*$ of the dual numbers. We identify the dual set $2\mathbf{S}$ by the subset Π^2_{S} of 2G* numbers in the matrix Π^2_*, which is further identified by the set of the prime pairs $\Pi_{\mathrm{S}} \times \Pi_{\mathrm{S}}$.

Definition*: *Four primes ξ, η, ζ and ϕ form the prime quartet $(\xi, \eta; \phi, \zeta)$ of dependent primes $((\xi, \eta; \zeta), \phi)$ if $\phi = \hat{\wedge}(\xi, \eta, \zeta)$. Further we will use the notation $(\xi, \eta; \phi, \zeta) = ((\xi, \eta; \zeta), \phi)$ and the names "the prime quartet" or "the 4-prime dependence" or "a prime 3-primes representation".*

Corollary*: Duality condition, the quartet of dependent primes and a prime 3-primes representation are equivalent. Exactly, for the prime pairs (ξ,η) and $[\phi,\zeta]$, $\phi>\xi,\eta,\zeta$

$$\hat{\wedge}(\xi,\eta)=\hat{\vee}(\phi,\zeta) \;\Leftrightarrow\; (\xi,\eta;\phi,\zeta)=((\xi,\eta;\zeta),\phi) \;\Leftrightarrow\; \phi=\hat{\wedge}(\xi,\eta,\zeta). \tag{13}$$

□ Duality of the pairs (ξ,η) and $[\phi,\zeta]$ implies

$$\hat{\wedge}(\xi,\eta)=\hat{\vee}(\phi,\zeta) \;\Leftrightarrow\; \xi+\eta=\phi-\zeta \;\Rightarrow\; (\xi,\eta;\phi,\zeta)=((\xi,\eta;\zeta),\phi) \;\Rightarrow\; \phi=\hat{\wedge}(\xi,\eta,\zeta).$$

Hence duality is equivalent to the 4-prime dependence or to a prime 3-primes representation.
By the weak Goldbach's conjecture for every prime ϕ three is a pair $(\phi,\ (\xi,\eta,\zeta))$ such that

$$\phi=\hat{\wedge}(\xi,\eta,\zeta) \;\Rightarrow\; (\xi,\eta;\phi,\zeta)=((\xi,\eta;\zeta),\phi) \;\Rightarrow \hat{\wedge}(\xi,\eta)=\hat{\vee}(\phi,\zeta).$$

Hence a prime 3-primes representation is equivalent to the 4-prime dependence or to the pairs duality.
In the followin example we have for

$$\begin{array}{ll}\phi=3: & ((1,1;1),3) \;\Leftrightarrow\; 3=\hat{\wedge}(1,1,1) \;\Leftrightarrow\; \hat{\wedge}(1,1)=2=\hat{\vee}(3,1)\\ \phi=5: & ((1,1;3),5) \;\Leftrightarrow\; 5=\hat{\wedge}(1,1,3)\\ \quad\Leftrightarrow & \hat{\wedge}(1,1)=2=\hat{\vee}(5,3),\ \hat{\wedge}(1,3)=4=\hat{\vee}(5,1).\end{array}$$

□

Corollary: *(L. Gerstein)* An even integer is sum of two primes, $2n=\hat{\wedge}(\xi,\eta)$, if and only if their conjugation function $f(n)=n^2-\tilde{n}^2=\xi\eta$.

□ Let $2n=\hat{\wedge}(\eta,\xi)$ is a 2G number. Consequently there is conjugated pair $(n,\tilde{n})$ which defines 2G* number $2\tilde{n}=\hat{\vee}(\eta,\xi)$. Since

$$n-\tilde{n}=\xi,\ n+\tilde{n}=\eta \;\Rightarrow\; n^2-\tilde{n}^2=\xi\eta=f(n).$$

Else, if there is a conjugated function $f(n)=n^2-\tilde{n}^2=\xi\eta$ then $n^2-\tilde{n}^2=(n+\tilde{n})(n-\tilde{n})=\xi\eta$, which defines $\xi=n+\tilde{n}$ and $\xi=n-\tilde{n}$ conjugated numbers. This implies directly $2n=\hat{\wedge}(\xi,\eta)$. □

Reconstruction of the Primes

The set of primes Π is the subset of all 3Π numbers, and by the Goldbach's weak conjecture may be reconstructed as the prime subset of the wedging of all possible prime triplets. Accordingly, the wedge operation maps $\Pi\times\Pi\times\Pi$ onto set Π^3 of all possible odd numbers which contains the prime numbers as a subset. Further, the prime projection operator $\hat{\Pi}$ is used to select all primes from there. Hence the following mapping diagram representing recreation of the prime numbers holds

$$\Pi \xrightarrow{\times} \Pi\times\Pi\times\Pi \xrightarrow{\times} \hat{\wedge}\,\Pi^3 \xrightarrow{\hat{\Pi}} \Pi \;\Leftrightarrow\; \Pi^2\times\Pi \xrightarrow{\times} \hat{\wedge}\,\Pi^3 \xrightarrow{\hat{\Pi}} \Pi. \tag{14}$$

We notice that the duality between 2G and 2G* numbers exhibited at the dual set 2**S** of the prime pairs necessarily creates the set Π_s of the primes supporting the duality. Exactly

$$\Pi_s=\{\phi:\exists(\xi,\eta;\zeta)\ \therefore\ \hat{\wedge}(\xi,\eta)=\hat{\vee}(\zeta,\phi) \;\Leftrightarrow\; (\xi,\eta;\phi,\zeta)=((\xi,\eta;\zeta),\phi) \;\Leftrightarrow\; \phi=\hat{\wedge}(\xi,\eta,\zeta)\}$$

We want to see if there is any substantial relation between the prime sets Π_s and Π.

Corollary*: The primes created on the set $2\mathbf{S}$ of all duals are all possible primes; the prime sets Π_s and Π are identical.

□ The duality condition, four prime quartet or four prime dependence and 3-primes representation are equivalent. Hence by definition of the set $\Pi_{\mathbf{s}}$

$$\Pi_s = \{\phi : \exists(\xi,\eta;\zeta) \ \therefore \ \hat{\wedge}(\xi,\eta) = \hat{\vee}(\zeta,\phi) \ \Leftrightarrow \ \phi = \hat{\wedge}(\xi,\eta,\zeta)\} \subset \Pi.$$

The weak Goldbach's conjecture is equivalent to four prime dependence and

$$\Pi = \{\phi : \phi = \hat{\wedge}(\xi,\eta,\zeta) \Rightarrow \hat{\wedge}(\xi,\eta) = \hat{\vee}(\zeta,\phi)\} \subset \Pi_{s}.$$

Hence the prime sets Π_s and Π are identical. Consequently, the primes created on the set of duals $2\mathbf{S}$ are all possible existing primes. □

Corollary*: All primes Π are the subset of the 3Π numbers created on the collection of all possible prime pairs by the prime lift operators $\hat{\zeta}$

$$\hat{\zeta} : (\xi,\eta) \to \zeta + \hat{\wedge}(\xi,\eta) = \phi, \ \ \forall\,(\xi,\eta) \in \ \Pi \times \Pi.$$

□ According to the Goldbach's weak conjecture all primes are created in the $\hat{1}\otimes\hat{1}\otimes\hat{1}$ prime bases and each prime is three primes dependent. However, a prime 3-primes representation and the duality are equivalent so that

$$\phi = \hat{\wedge}(\xi,\eta,\zeta) \ \Rightarrow \ \hat{\wedge}(\xi,\eta) = \hat{\vee}(\phi,\zeta), \ \ \xi,\eta,\zeta < \phi.$$

Hence, there is the prime pair dependent operator $\hat{\Phi}_{\xi,\eta}$ creating the prime ϕ in the row $A_{\xi\eta}$

$$\hat{\Phi}_{\xi,\eta}(\Pi) = \{\phi : \ \exists\zeta \in \Pi \ \therefore \ \hat{\wedge}(\xi,\eta) = \hat{\vee}(\phi,\zeta)\}.$$

and instead in the $\hat{1}\otimes\hat{1}\otimes\hat{1}$ prime bases the primes are recreated in the $\hat{2}\otimes\hat{1}$ bases of the $(\Pi\times\Pi)\times\Pi$ set. Exactly, for each prime ϕ there is a pair (ξ,η) in $\Pi\times\Pi$ and a prime $\zeta \in \Pi$ creating the pair $(\phi,\zeta) \in \Pi_*\times\Pi_*$ so that ϕ is the lift

$$\hat{\zeta}(\xi,\eta) = \zeta + \hat{\wedge}(\xi,\eta).$$

of the pair (ξ,η) by the prime $\zeta \in \Pi$. □

Remark: The prime bases $\hat{2}\otimes\hat{1}$ is naturally suited for the Π^3 multiplication matrix. For, the matrix is spanned by the row axes $\mathcal{R}$ of the primes ζ and by the column axes $\mathcal{D}$ of al possible prime pairs (ξ,η), collected into all proper prime families $\Xi(\xi) = \{(\xi,\eta) : \eta = 1,3,\cdots,\xi\}$, ordered by the primes ξ. Every prime pair (ξ,η) creates an infinite row

$$\Pi(\xi,\eta) = \{\zeta + \hat{\wedge}(\xi,\eta) : \zeta \in \Pi\} \subset \Pi.$$

of distinct prime species in an increasing order in the multiplication matrix Π^3.

Definition: *The operator $\hat{\Pi}$ is the prime projector selecting all primes from the multiplication matrix set Π^3. Its subset $\hat{\Pi}_{\xi\eta} = \{\hat{\zeta}(\xi,\eta) : \zeta \in \Pi\}$ of the distinct prime projection operators selects all primes from the row $A_{\xi\eta}$.*

The prime ζ variable takes values from the prime axes $\mathcal{R}$ so that each selected prime $\phi = \phi_\zeta$ is really the prime selector ζ dependent. Hence the duality condition

$$\hat{\wedge}(\xi,\eta) = \hat{\vee}(\phi,\Pi_p)$$

relates all selected prime family $\Pi(\xi,\eta)$ to a single prime pair (ξ,η).
Equivalent pairs, represented by a single pair $\overline{(\xi,\eta)}$, create identical families of the selected primes represented by a single representative $\overline{\Pi(\xi,\eta)}$. Notice that a proper family $\Xi(\xi)$ creates a strip $\{\Pi(\xi,\eta)\}_1^{\xi}$ of distinct prime rows. Finally

$$\Pi = \bigcup \{\phi\} = \bigcup \Pi(\xi,\eta) = \bigcup \overline{\Pi(\xi,\eta)}$$

are all existing primes. There is no primes other then the primes recreated on this way.

The following table shows the upper triangular part of the prime recreation matrix. The bold column numbers are the prime enumeration of the strips and the bold integers in the prime recreation matrix are the largest entries $p+p+p$ in each p-prime family.

Table 5. The prime matrix Π^3

ξ	η	$\hat{\wedge}(\xi,\eta)$	$\zeta \rightarrow$	1	3	5	7	11	13	17	19	23	$*$
1	1	2		**3**	5	7		13		19			$*$
3	1	4			7		11	17			23		$*$
	3	6			**9**	11	13	17	19	23		29	$*$
5	1	6				11	13	17	19	23		29	$*$
	3	8				13		19				31	$*$
	5	10				**15**	17		23		29		$*$
7	1	8						19				31	$*$
	3	10					17		23		29		$*$
	5	12					19	23		29	31		$*$
	7	14					**21**			31		37	$*$
11	1	12						23		29	31		$*$

The Trinity Equation

We notice that the variable $\mathcal{D}$ of all pairs on the column axes of the multiplication matrix $\Pi\times\Pi\times\Pi$ and its conjugated variable $\mathcal{D}^*$ of all anti-wedging pairs are implicitly related in the multiplication matrix $(\Pi\times\Pi)\times\Pi$. For, a prime $\phi \in \mathcal{P}^*$ is in the rage of the wedge function, each pair (ξ,η) from $\mathcal{D}$ is lifted to the ϕ in the row $\mathrm{A}(\xi,\eta)$ by a prime ζ lifting operator, so that $[\phi,\zeta]$ is in $\mathcal{D}^*$.

Further, all 3G numbers in the row $\mathrm{A}(\xi,\eta)$ of a prime pair (ξ,η) are divided into set of the primes $\Pi(\xi,\eta)$ and the complementary set $\mathrm{A}(\xi,\eta)\setminus\Pi(\xi,\eta) = \Pi^c(\xi,\eta)$ of the non-prime numbers. This necessarily partitions the sets of the primes projectors into projectors $\hat{\Pi}_p$ of the primes and the complementary set $\hat{\Pi}_p^c$ of the projectors of the non-prime 3G numbers.

Corollary*: The sets $\mathcal{D}$ and $\mathcal{D}^*$ are mutually dual. The duality is exclusively set on each row $\mathrm{A}(\xi,\eta)$ of the multiplication matrix $\Pi^2\times\Pi$ by the mutual duality of the wedge prime pair (ξ,η) and an arbitrary anti-wedge pair $[\phi,p] \in \mathcal{D}^*$, $\hat{\vee}(\phi,p) = \hat{\wedge}(\xi,\eta)$, and induced on an arbitrary 3G number pair $[3a,\zeta] \in \mathcal{D}^*$ on the same row by

$$\hat{\vee}(3a,\zeta) = \hat{\wedge}(\xi,\eta) = \hat{\vee}(\phi,p). \tag{15}$$

$\square$ All 3G numbers created on the pair (ξ, η) by a lift operator $\hat{\zeta} : \hat{\zeta}(\xi, \eta) = \zeta + \hat{\wedge}(\xi, \eta)$, with $\zeta \in \Pi$ are held in the row $\mathrm{A}(\xi, \eta)$ set. All primes created there are $\phi, \psi, \Phi, \Psi, \cdots \in \Pi(\xi, \eta)$, and all non-prime created 3G numbers are $3b$ in the $\Pi^{\mathrm{c}}(\xi, \eta)$ set. Thus $\mathrm{A}(\xi, \eta) = \Pi(\xi, \eta) \cup \Pi^{\mathrm{c}}(\xi, \eta)$.

$\zeta \rightarrow$	$\cdots$	p	θ_1	θ_2	$\cdots \theta_{\mathrm{N}}$	q	$\cdots$
$\hat{\wedge}(\xi, \eta)$	$\cdots$	ϕ_p	b_1	b_2	$\cdots b_{\mathrm{N}}$	ϕ_q	$\cdots$

Let ϕ_p and ϕ_q are two consecutive primes created by two prime projectors $\hat{p}$ and $\hat{q}$ from the set $\hat{\Pi}_p$. The primes p and q, $p < q$, are placed on the prime axes in the environment of the natural numbers in the increasing order and the mutual distances g_p^q fixed by the natural prime generating machine. Consequently, the primes ϕ_p and ϕ_q are created by the operator $\hat{\Phi}_{(\xi,\eta)}$ in the row $\mathrm{A}(\xi, \eta)$ in the order and mutual distance prescribed by the primes p and q. Hence

$$\hat{\Phi}_{(\xi,\eta)}(q) = \hat{\Phi}_{(\xi,\eta)}(p) + \hat{\vee}(\phi_q \phi_p) \quad \therefore \quad \phi_q = \phi_p + g_{\phi_p}^{\phi_q},$$

and the distance between primes ϕ_p and ϕ_q

$$\hat{\vee}(\phi_q, \phi_p) = g_{\phi_p}^{\phi_q} = p + \hat{\wedge}(\xi, \eta) - (q + \hat{\wedge}(\xi, \eta)) = \hat{\vee}(q, p) = g_p^q,$$

is the same as the distance between the primes p and q. The number of the primes on the prime segment $[p \cdots q]$ on the $\mathcal{R}$ axes is calculated by the prime counting function π, and $\mathrm{N} = \pi(q) - \pi(\eta)$. Hence there is a sequence $\theta_1, \theta_2, \cdots, \theta_{\mathrm{N}}$ of N primes between the primes p and q in the Π_p^{c} prime set creating the lift sequence $3b_1, 3b_2, \cdots, 3b_{\mathrm{N}}$ of non-prime numbers in the row $\mathrm{A}^{\mathrm{c}}(\xi, \eta)$. Since the row $\mathrm{A}(\xi, \eta)$ is ordered set all sequence is created between the primes ϕ_p and ϕ_q. In altogether, the created sequence on the closed segment $[\phi_p \cdots \phi_q]$ is $\{\phi_p,\ 3b_1, \cdots, 3b_{\mathrm{N}},\ \phi_q\}$ so that

$$\begin{array}{llll}
\phi_p & = \hat{\wedge}(\xi, \eta) + p & \Leftrightarrow & \hat{\wedge}(\xi, \eta) = \hat{\vee}(\phi_p, p), \\
3b_1 & = \hat{\wedge}(\xi, \eta) + \theta_1 & \Leftrightarrow & \hat{\wedge}(\xi, \eta) = \hat{\vee}(3b_1, \theta_1), \\
\cdots & \cdots\cdots\cdots\cdots\cdots & & \cdots\cdots\cdots\cdots\cdots\cdots \\
3b_{\mathrm{N}} & = \hat{\wedge}(\xi, \eta) + \theta_{\mathrm{N}} & \Leftrightarrow & \hat{\wedge}(\xi, \eta) = \hat{\vee}(3b_{\mathrm{N}}, \theta_{\mathrm{N}}), \\
\phi_q & = \hat{\wedge}(\xi, \eta) + q & \Leftrightarrow & \hat{\wedge}(\xi, \eta) = \hat{\vee}(\phi_p, p),
\end{array}$$

The last shows that the following duality relations hold

$$\hat{\wedge}(\xi, \eta) = \hat{\vee}(\phi_p, p) \quad = \hat{\vee}(3b_1 \theta_1) = \cdots = \hat{\vee}(3b_{\mathrm{N}}, \theta_{\mathrm{N}}) = \quad \hat{\vee}(\phi_q, q).$$

Hence the fundamental duality relation on each row $\mathrm{A}(\xi, \eta)$ is exclusively imposed by the primes and induced on all non-prime 3G numbers there. Exactly, the prime pair (ξ, η) is dual to all prime pairs $[\phi, p]$ in the prime created set $\Pi(\xi, \eta)$, and induced on all non-prime $[3b, \theta]$ pairs from the $\mathrm{A}^{\mathrm{c}}(\xi, \eta)$ set. Hence the prime pair (ξ, η) is dual to all $[3\mathrm{A}, \zeta]$ pairs on the row $\mathrm{A}(\xi, \eta)$ and

$$\hat{\wedge}(\xi, \eta) \quad = \hat{\vee}(\phi, p) = \hat{\vee}(3b, \theta) = \ \hat{\vee}(3a, \zeta).$$

Notice that $\hat{\vee}(3b, \theta) = \hat{\vee}(3a, \zeta)$ is the equivalence condition "on all of the anti-wedge pairs of the conjugated $\mathcal{D}^* \cap \mathrm{A}(\xi, \eta)$ set ". Since created primes and non-primes are all 3G numbers in the row $\mathrm{A}(\xi, \eta)$ and since the row is an arbitrary row

$$\forall(3a), \exists((\xi, \eta; \zeta), \phi) \quad \therefore \quad \hat{\vee}(3a, \zeta) = \hat{\wedge}(\xi, \eta) = \hat{\vee}(\phi, p).$$

Finally, all prime pairs (ξ, η) are in $\mathcal{D}$ and all anti-wedge prime pairs $[3a, \zeta]$ in $\mathcal{D}^*$ and the sets $\mathcal{D}$ and $\mathcal{D}^*$ are mutually dual. $\square$

Goldbach's Strong Conjecture

Let $2x$ be an even, not necessary 2G number. Its associated odd integer $3X = 2x+1$ is a 3G number by the weak Goldbach's conjecture and has a 3-primes representation $3X = \hat{\wedge}(\alpha, \beta, \gamma)$. Further

$$2X = 3X + 1 = 2x + 2 = \hat{\wedge}(\alpha, \beta, \gamma, 1).$$

is an even number natural companion of the $3X$ number. The collection of all 3G numbers and their companions $3\text{G} + 1$ numbers are in one to one correspondence. Further

$$\begin{aligned} 2X &= \hat{\wedge}(\alpha, \beta) + \hat{\wedge}(\gamma, 1) && \{\exists\ \phi \in \Pi_p, \exists \theta \in \Pi_p :\ (\phi, \theta) \in \mathcal{D}^*\}\ \therefore \\ &= \hat{\wedge}(\alpha, \beta) + \hat{\vee}(\phi, \theta) && \\ &= \hat{\wedge}(\alpha, \beta, \phi) - \theta && \{\exists (3a) \in 3\mathcal{P} : 3a = \hat{\wedge}(\alpha, \beta, \phi)\}\ \therefore \\ &= \hat{\vee}(3a, \theta) && \end{aligned}$$

If $3a = \Psi$ is the prime number then $2X = \hat{\vee}(\Psi, \theta) \equiv \hat{\wedge}(\xi, \eta)$ by the prime pairs duality relation. Else, the general case is treated by the application of "the trinity equation" identities. Thus there are $\varphi \in \Pi_p$, and $\Phi \in \Pi_p$ such that the pair $(\Phi, \phi) \in \mathcal{D}^*$ and

$$2X = \hat{\vee}(3a, \theta) = \hat{\vee}(\Phi, \varphi) = \hat{\wedge}(\xi, \eta).$$

Conclusion

The conclusion is given in the following summarizing corollary.

Corollary*: All natural numbers are the Goldbach's numbers.

□ It is established by the work of Harald A. Helfgott, N.Vinogradov and others that every odd natural number is a sum of three primes, and therefore the Goldbach's number. Its even companion is the sum of two primes. That property stems from the fundamental nature of the prime numbers and necessarily implies that every even natural number is composed of two primes. Thus, all natural numbers are the Goldbach's numbers. □

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