SOLUTION OF BROCARD‘S PROBLEM 20.12.2017

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Abstract. In this article I will provide the solution of Brocard‘s problem and I will prove the existence of the finite amount of Brown numbers, where the largest Brown number is $(7, 71)$, which represents the equation $7! + 1 = 71^2$.

1. Introduction

Brocard‘s problem represents one of the open problem in mathematics from the field of number theory, which has been formulated by Henri Brocard in 1876 and represents the solutions of Diophantine equation

$$n! + 1 = x^2 \tag{1.1}$$

The main object of this problem is to prove or disprove the existence of number $n > 7$ for which the equation holds, equivalently the existence of only the three Brown numbers $(4, 5)$, $(5, 11)$ a $(7, 11)$.

2. Notations

Brown number is a number in the form (n, x), where $n, x \in \mathbb{N}$ and for which the equation (1.1) holds. Notation in the form $z[.k]$, where $z, k \in \mathbb{N}_0$ denotes the digit k in the one‘s place, notation in the form $z[.k], [.k_1], ...[.k_x]$ denotes the digits k which can be in the one‘s place of number z.

Example 2.1. $127[.7]$ *represents the number* 127

Let number $Z \in \mathbb{N}$. Let $2x_n$ denotes an even number $\leq n$. $\prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n)$ represents the product of all even numbers $2x_n \leq n$. Let $\Delta_p(n)$ denotes the power of prime number p, such that

$$p^{\Delta_p(n)} = \frac{\prod_{x_p=1}^{\lfloor \frac{n+p}{2p} \rfloor}(p * (2x_p - 1))}{\prod_{x=1}^{\lfloor \frac{n+p}{2p} \rfloor}(2x - 1)}$$

if $n!|p \wedge (2x - 1) \neq p^Z$.

Example 2.2. $\Delta_3(14) = 3$, $\Delta_5(26) = 4$

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3. Elementary attributes of the equation

Lemma 3.1. $\forall(n > 4), (n! + 1)[.1]$

Proof. Since in the prime decomposition of $n!$ the prime numbers 2 and 5 makes $n!$ divisible by number 10 and in order to $n!|10$ must be according to the divisibility rules by number 10 $\forall(n > 4)$ Lemma 3.1 truth, on the basis of the fact that the sum in the one‘s place is $0 + 1 = 1$. □

Lemma 3.2. $\forall(n > 4), (n! + 1)[.1]|x^2 \leftrightarrow x[.1], [.9]$

Proof. From the equation (1.1) rearranged to the form

$$\sqrt{(n! + 1)} = x$$

follows, that x must be only in the form $x[.1], [.9]$ on the basis of the elementary rule of arithmetic regarding the fact, that only the numbers 1 and 9 in the one‘s place raised to the power of two can make the number in the form of $(n! + 1)[.1]$. □

4. Solution

From Lemma 3.2 is clear, that x is an odd number an we can express it as $2x_2 + 1$ and we can rearrange the equation (1.1) to the form

$$n! + 1 = (2x_2 + 1)^2 \tag{4.1}$$

and at the same time from Lemma 3.2 follows, that $(2x_2 + 1)$ must be only in the form $(2x_2 + 1)[.1], [.9]$.

Corollary 4.1. From Lemma 3.2 follows, that x is an odd number in the form $(2x_2 + 1)$ and there exist no number $(Z > 2)$ respectively the odd number in the form of $(Zx_2 + 1)$ for which the truth of each Theorem would be disproved. The main goal is to refute the ambiguity of the truth of each Theorem, since their proves are directly based on the odd number in the form of $(2x_2 + 1)$.

Proof. Since by the expression $(2x_2 + 1)$ can be expressed all odd numbers in the set of integers and mainly $\forall(Z > 2)$ holds, that $(Zx_2 + 1)$ is a subset of odd numbers to $(2x_2 + 1)$, what would be in the contradiction, which we can express as follows

$$(2x_2 + 1) \leftrightarrow Theorems \land (Zx_2 + 1) \not\leftrightarrow Theorems$$

which can never happen, because $\forall(Z > 2), (Zx_2 + 1) \subset (2x_2 + 1)$. □

4.1. x in the form [.1].

Lemma 4.2. $x = (2x_2 + 1) \rightarrow (2x_2 + 1)[.1] \leftrightarrow x_2[.5], [.0]$

Proof. After assigning the number $(5 + 10Z)[.5]$ in the expression $(2x_2 + 1)$ instead of x_2 the expression $(2x_2 + 1)$ will be as follows

$$(2 * (5 + 10Z) + 1) = (11 + 20Z)$$

and if we assign instead of x_2 the number in the form $(10Z)[.0]$, the expression $(2x_2 + 1)$ will be in the form

$$(2 * (10Z) + 1) = (1 + 20Z)$$

from that follows, that the expression $(2x_2 + 1)$ will be in the form $(2x_2 + 1)[.1]$ if and only if x_2 will be in the form $x_2[.5], [.0]$. □

After assigning $x_{2[.0]}$ in the equation (4.1), the equation will be as follows

$$n! + 1 = (2x_{2[.0]} + 1)^2 \tag{4.2}$$

After rearrangement of equation (4.2) we get

$$n! + 1 = 4x_{2[.0]}^2 + 4x_{2[.0]} + 1 \tag{4.3}$$

$$n! = 4x_{2[.0]}^2 + 4x_{2[.0]} \tag{4.4}$$

$$\frac{n!}{4x_{2[.0]}} = x_{2[.0]} + 1 \tag{4.5}$$

Lemma 4.3. *According to Lemma 3.1* $\forall(n > 4), n![.0]$ *from that follows, that if RHS of equation (4.5) is in this case in the form* $(x_{2[.0]} + 1)[.1]$ *(in the one‘s place is sum* $0 + 1 = 1$*)* $4x_{2[.0]}$ *must divide all even numbers* $2x_n \leq n$ *and* $5^{\Delta_5(n)}$*, and therefore the minimal form of* $4x_{2[.0]}$ *must be*

$$4x_{2[.0]} = \prod_{x_n=1}^{\lfloor n/2 \rfloor} (2x_n) * 5^{\Delta_5(n)} \tag{4.6}$$

Proof. According to the elementary rules of arithmetic a number Z is an odd number only in that case when in the one‘s place is not an even number (including zero), in other words in the prime decomposition is not prime number 2. According to the divisibility rules $Z|5 \leftrightarrow Z[.0], [.5]$, and therefore must be $5^{\Delta_5(n)}$ divided too, otherwise the quotient would be in the form $\frac{n!}{4x_{2[.0]}}[.0]$ and without even numbers $\frac{n!}{4x_{2[.0]}}[.5]$. □

Theorem 4.4. *From Lemma 4.3 and from rearrangement of the equation (4.6) follows, that* $x_{2[.0]}$ *must be in the minimal form*

$$x_{2[.0]} = \frac{\prod_{x_n=1}^{\lfloor n/2 \rfloor} (2x_n) * 5^{\Delta_5(n)}}{4} \tag{4.7}$$

then $\forall(n > 4)$!

$$x_{2[.0]} > x_2 \tag{4.8}$$

what is in the contradiction with fact, that

$$x_{2[.0]} = x_2 \tag{4.9}$$

Proof. From equation (4.1) by expressing x_2 we get

$$x_2 = \frac{\sqrt{n! + 1} - 1}{2} \tag{4.10}$$

For factorial in the form $(2n)!$ holds following

$$(2n)! = \prod_{x_n=1}^{n} (2x_n) * \prod_{x_n=1}^{n} (2x_n - 1) \tag{4.11}$$

where

$$\prod_{x_n=1}^{n} (2x_n) = \prod_{k=0}^{n-1} (2n - 2k) \tag{4.12}$$

and

$$\prod_{x_n=1}^{n}(2x_n-1)=\prod_{k=0}^{n-1}(2n-2k-1) \tag{4.13}$$

then

$$\frac{\prod_{k=0}^{n-1}(2n-2k)}{\prod_{k=0}^{n-1}(2n-2k-1)}>1 \tag{4.14}$$

from which follows, that

$$\sqrt{\frac{\prod_{k=0}^{n-1}(2n-2k)}{\prod_{k=0}^{n-1}(2n-2k-1)}}>1 \tag{4.15}$$

and at the same time holds, that after rearrangement of

$$\sqrt{(2n)!}=\sqrt{\prod_{x_n=1}^{n}(2x_n)*\prod_{x_n=1}^{n}(2x_n-1)} \tag{4.16}$$

we get inequality

$$\sqrt{\prod_{x_n=1}^{n}(2x_n)}>\sqrt{\prod_{x_n=1}^{n}(2x_n-1)} \tag{4.17}$$

The inequality (4.17) is equivalent of the inequality (4.15). Since the quotient in the inequality (4.15) is only increasing and

$$\lim_{n\to\infty}\sqrt{\frac{\prod_{k=0}^{n-1}(2n-2k)}{\prod_{k=0}^{n-1}(2n-2k-1)}}=\infty \tag{4.18}$$

$\forall(n>4)$ respectively $\forall(n>9)!$ holds the following inequality

$$\frac{\prod_{x_n=1}^{n}(2x_n)}{4}>\frac{\sqrt{(2n)!+1}-1}{2} \tag{4.19}$$

and at the same time must hold the following inequality

$$x_{2[.0]}=\frac{\prod_{x_n=1}^{n}(2x_n)*5^{\Delta_5(2n)}}{4}>x_2=\frac{\sqrt{(2n)!+1}-1}{2} \tag{4.20}$$

since

$$5^{\Delta_5(2n)}>1 \tag{4.21}$$

$\forall(n>2)$ respectively $\forall(n>4)!$, and therefore the inequality (4.8) holds $\forall(n>4)!$ for factorial in the form $(2n)!$. For factorial in the form $(2n+1)!$, which we can write as $(2n)!*(2n+1)$ and after assigning it in the inequality (4.20), the inequality will be in the form

$$x_{2[.0]}=\frac{\prod_{x_n=1}^{n}(2x_n)*5^{\Delta_5(2n)}}{4}>x_2=\frac{\sqrt{((2n)!*(2n+1))+1}-1}{2} \tag{4.22}$$

where for better imagination we can modify the inequality by neglecting the terms of expression which do not have any influence on the proof needed for proving Theorem (4.4) to the form

$$\frac{\prod_{x_n=1}^{n}(2x_n) * 5^{\Delta_5(2n)}}{4} > \frac{\sqrt{(2n)!}}{2} * \sqrt{(2n+1)} \tag{4.23}$$

what clearly proves, that inequality (4.22) holds, because from inequality (4.19) is clear, that $\forall(n > 9)!$ holds

$$\frac{\prod_{x_n=1}^{n}(2x_n)}{4} > \frac{\sqrt{(2n)!+1}-1}{2} \approx \frac{\sqrt{(2n)!}}{2} \tag{4.24}$$

and regarding the fact, that $\forall(n > 2)$ respectively $\forall(n > 4)!$ holds

$$5^{\Delta_5(2n)} > \sqrt{(2n+1)} \tag{4.25}$$

Example 4.1. $5^{\Delta_5(6)} > \sqrt{(2*3)+1}$

must hold inequality (4.22) as well. From the mentioned facts follows, that $\forall(n > 4)!$ holds inequality (4.8) for factorial in the form $(2n+1)!$ and for factorial in the form $(2n)!$ and that proves the Theorem (4.4). □

After assigning $x_{2[.5]}$ in the equation (4.1), the equation will be as follows

$$n! + 1 = (2x_{2[.5]} + 1)^2 \tag{4.26}$$

After rearrangement of the equation (4.26) we get

$$n! + 1 = 4x_{2[.5]}^2 + 4x_{2[.5]} + 1 \tag{4.27}$$

$$n! = 4x_{2[.5]}^2 + 4x_{2[.5]} \tag{4.28}$$

$$\frac{n!}{4x_{2[.5]} + 4} = x_{2[.5]} \tag{4.29}$$

Theorem 4.5. *From the fact, that $x_{2[.5]}$ must be only in the form*

$$x_{2[.5]}[.5] \tag{4.30}$$

from the equation (4.29) follows, that $\forall(n > 9)!$

$$x_{2[.5]}[.9] \tag{4.31}$$

what is in contradiction with fact, that $x_{2[.5]}$ must be only in the form (4.30).

Proof. Since according to Lemma 3.1 $\forall(n > 4), n![.0]$ and according to the elementary rules of arithmetic only after multiplication of an odd number by number 5 can the number Z be in the form $Z[.5]$, and therefore $n!$ after division by $(4x_{2[.5]} + 4)$ may not have in the prime decomposition prime number 2. From that follows, that $(4x_{2[.5]} + 4)$ must divide all even numbers $2x_n \leq n$ and must be in the minimal form

$$(4x_{2[.5]} + 4) = \prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n) \tag{4.32}$$

Where after rearrangement we will express $x_{2[.5]}$ as follows

$$x_{2[.5]} = \frac{\prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n)}{4} - 1 \tag{4.33}$$

From the equation (4.33) follows, that $\forall(n>9)!$ $x_{2[.5]}$ is only in the form (4.31), because $\forall(n>9)!$ $\frac{\prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n)}{4}$ is in the form

$$\frac{\prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n)}{4}[.0] \tag{4.34}$$

since the product $\prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n)$ will include number 10, and therefore $\forall(n>9)!$ will be $\frac{\prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n)}{4}$ in the form (4.34) and the difference in the one's place (0-1) in the equation (4.33) will cause, that $x_{2[.5]}$ will be only in the form (4.31), what clearly proves Theorem (4.5). From the Theorem (4.4) and (4.5) follows, that x_2 can not be in the form $x_2[.0],[.5]$ and on the basis of this fact there exists no x in the form $(2x_2+1)[.1]$, and therefore equation (4.1) if $(2x_2+1)[.1]$ does not have any solution $\forall(n>9)!$, since for the case when x_2 was in the form $x_{2[.5]}$ I have proved the Theorem $\forall(n>9)!$. □

4.2. x in the form [.9].

Lemma 4.6. $x=(2x_2+1)\rightarrow(2x_2+1)[.9]\leftrightarrow x_2[.4],[.9]$.

Proof. After assigning the number $(4+10Z)[.4]$ in the expression $(2x_2+1)$ instead of x_2 the expression $(2x_2+1)$ will be as follows

$$(2*(4+10Z)+1)=(9+20Z)$$

if we assign instead of x_2 the number in the form $(9+10Z)[.9]$, the expression $(2x_2+1)$ will be in the form

$$(2*(9+10Z)+1)=(19+20Z)$$

from that follows, that expression $(2x_2+1)$ will be in the form $(2x_2+1)[.9]$ if and only if x_2 will be in the form $x_2[.4],[.9]$. □

After assigning $x_{2[.4]}$ in the equation (4.1), the equation will be as follows

$$n!+1=(2x_{2[.4]}+1)^2 \tag{4.35}$$

after rearrangement

$$n!+1=4x_{2[.4]}^2+4x_{2[.4]}+1 \tag{4.36}$$

$$n!=4x_{2[.4]}^2+4x_{2[.4]} \tag{4.37}$$

$$\frac{n!}{4x_{2[.4]}}=x_{2[.4]}+1 \tag{4.38}$$

Theorem 4.7. *From the equation (4.38) follows, that*

$$x_{2[.4]}=\frac{\prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n)}{4} \tag{4.39}$$

then $\forall(n>9)!$

$$x_{2[.4]}[.0] \tag{4.40}$$

what is in contradiction with fact, that $x_{2[.4]}$ *must be only in the form*

$$x_{2[.4]}[.4] \tag{4.41}$$

Proof. According to Lemma 3.1 $\forall(n>4), n![.0]$ and at the same time from that follows, that if RHS of equation (4.38) is in this case in the form $(x_{2[.4]}+1)[.5]$ (in the one's place is the sum 4+1=5) the expression $4x_{2[.4]}$ must divide all even numbers $2x_n \leq n$, and therefore $4x_{2[.4]}$ must be in the minimal form

$$4x_{2[.4]} = \prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n) \tag{4.42}$$

after expressing of $x_{2[.4]}$ equation (4.42) will be in the form

$$x_{2[.4]} = \frac{\prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n)}{4} \tag{4.43}$$

Where $\forall(n>9)!$ will be $x_{2[.4]}$ in the form (4.40), because the product $\prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n)$ will include number 10, and therefore $\forall(n>9)!$ will be $x_{2[.4]} = \frac{\prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n)}{4}$ in the form

$$\frac{\prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n)}{4}[.0] \tag{4.44}$$

what is in contradiction with fact, that $x_{2[.4]}$ must be only in the form (4.41) and at the same time it proves the Theorem (4.7). □

Corollary 4.8. From the Theorem (4.7) respectively from the equation (4.39) follows, that at minimum the $4x_{2[.4]}$ must be equal to $4x_{2[.4]} = \prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n)$. There exists no $k \in \mathbb{Z}$ for which would after assigning in the equation (4.39) the following equation holds

$$4x_{2[.4]} = \prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n) + k \tag{4.45}$$

and at the same time disprove the statement (4.40), which follows from the Theorem (4.7).

Proof. Still must be true the statement, that $4x_{2[.4]}$ must divide at minimum all even numbers $2x_n \leq n$ from $n!$, and therefore if there exists a k the RHS of (4.45) must still satisfy this condition what is possible only in the case when

$$\prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n) + k = \prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n) * (2Z+1) \tag{4.46}$$

if $n!|(2Z+1)$. After assigning RHS of equation (4.46) in the equation (4.45) and by expressing $x_{2[.4]}$

$$x_{2[.4]} = \frac{\prod_{x_n=1}^{\lfloor n/2 \rfloor}(2x_n) * (2Z+1)}{4} \tag{4.47}$$

$\forall(n>9)!$ will be $x_{2[.4]}$ only in the form $x_{2[.4]}[.0]$ which follows from the Theorem (4.7), and therefore there exists no k which would hold for equation (4.45) and at the same time disprove the statement (4.40) and the Theorem (4.7). The form $x_{2[.4]}[.0]$ will be the same even in that case, when instead of one odd number $(2Z+1)$ would be the product of odd numbers. □

After assigning $x_{2[.9]}$ in the equation (4.1), the equation will be as follows

$$n! + 1 = (2x_{2[.9]} + 1)^2 \tag{4.48}$$

after elementary rearrangement

$$n! + 1 = 4x_{2[.9]}^2 + 4x_{2[.9]} + 1 \tag{4.49}$$

$$n! = 4x_{2[.9]}^2 + 4x_{2[.9]} \tag{4.50}$$

$$\frac{n!}{4x_{2[.9]} + 4} = x_{2[.9]} \tag{4.51}$$

According to Lemma 3.1 $\forall(n > 4), n![.0]$. From that follows, that if RHS of equation (4.51) is in this case in the form $(x_{2[.9]})[.9]$ (in the one‘s place is only the number 9) $(4x_{2[.9]} + 4)$ must divide all even numbers $2x_n \leq n$ and $5^{\Delta_5(n)}$, and therefore the minimal form of $4x_{2[.9]}$ must be

$$4x_{2[.9]} + 4 = \prod_{x_n=1}^{\lfloor n/2 \rfloor} (2x_n) * 5^{\Delta_5(n)} \tag{4.52}$$

Theorem 4.9. *From Lemma (4.3) and after rearrangement of the equation (4.52) follows, that $x_{2[.9]}$ is*

$$x_{2[.9]} = \frac{\prod_{x_n=1}^{\lfloor n/2 \rfloor} (2x_n) * 5^{\Delta_5(n)}}{4} - 1 \tag{4.53}$$

then $\forall(n > 4)!$

$$x_{2[.9]} > x_2 \tag{4.54}$$

what is in contradiction with fact, that

$$x_{2[.9]} = x_2 \tag{4.55}$$

Proof. For this case we can after small modifications apply the Theorem (4.4), but for better logical sequence of this article I will specify the proof more detailed. Regarding the fact, that from inequality (4.15) and from the definition of the limit of the expression (4.18) and at the same time according to inequality (4.20) and (4.21) is clear, that for factorial in the form $(2n)!$ must hold following inequality

$$x_{2[.9]} = \frac{\prod_{x_n=1}^{n} (2x_n) * 5^{\Delta_5(2n)}}{4} - 1 > x_2 = \frac{\sqrt{(2n)! + 1} - 1}{2} \approx \frac{\sqrt{(2n)!}}{2} \tag{4.56}$$

from that follows, that inequality (4.54) holds $\forall(n > 4)!$ for factorial in the form $(2n)!$. For factorial in the form $(2n+1)!$ expressed as $(2n)! * (2n+1)$ and by using the fact, that inequality (4.22) and (4.25) holds, is clear that must hold the following

$$x_{2[.9]} = \frac{\prod_{x_n=1}^{n} (2x_n) * 5^{\Delta_5(2n)}}{4} - 1 > x_2 = \frac{\sqrt{((2n)! * (2n+1)) + 1} - 1}{2} \tag{4.57}$$

From the mentioned facts follows, that $\forall(n > 4)!$ holds inequality (4.54) for factorial in the form $(2n)!$ and for factorial in the form $(2n + 1)!$ and at the same time it proves the Theorem (4.9). The rigorous proof of the Theorem (4.7) and (4.9) unconditionally supports the statement, that x_2 can not be in the form $x_2[.4], [.9]$ and on the basis of this fact there exists no x in the form $(2x_2 + 1)[.9]$, and therefore the equation (4.1) if

$(2x_2 + 1)[.9]$ does not have any solution $\forall(n > 9)!$, since for the case when x_2 was in the form $x_{2[.4]}$ I have proved the Theorem $\forall(n > 9)!$. □

5. Conclusion

The rigorous proof of the Theorem (4.4), (4.5) for x in the form $x[.1]$ and the Theorem (4.7), (4.9) for x in the form $x[.9]$ result in the final statement, that there exists no $n > 7$ for which the Diophantine equation $n! + 1 = x^2$ holds, and therefore there are only the three following Brown numbers $(4, 5), (5, 11), (7, 71)$.

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