The Lorentz Transformation from Light-Speed Invariance Alone

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Abstract

The derivation of the Lorentz transformation normally rests on two a priori demands, namely that reversing the direction of the transformation's constant-velocity boost inverts the transformation, and that the transformation leaves light-speed invariant. It is notable, however, that the simple light-clock concept, which is rooted entirely in light-speed invariance, immediately gives rise to special-relativistic reciprocal time dilation, whose existence implies that of special-relativistic length contraction as a corollary. Reciprocal time dilation and length contraction are key consequences of the Lorentz transformation, so it would seem that demanding only inertial transformation-invariance of light-speed already uniquely implies the Lorentz transformation. We find that to indeed be so, but it can't be shown by solving directly for the undetermined parameters of a general x-direction inertial space-time transformation; one necessarily must work with the corresponding velocity transformation (that has the same undetermined parameters), which is pared down to equalities that refer as predominantly as is feasible to speed. In those equalities both untransformed and transformed speed are replaced by the constant c, after which the undetermined parameters of the inertial transformation are solved for, which shows it to be the Lorentz transformation.

Introduction

The Lorentz transformation is normally derived on the basis of two a priori demands, namely that reversal of the direction of the transformation's constant-velocity boost inverts the transformation, and that the transformation leaves light-speed invariant. The Galilean transformation *shares* the "inversion by boost reversal" property of the Lorentz transformation, but the Galilean transformation leaves *time*, rather than light-speed, invariant.

The Galilean transformation which transforms the inertial frame of reference by the addition of the x-direction constant-velocity boost $(v, 0, 0)$ is,

$$(t',\, x',\, y',\, z') = (t,\, (x - vt),\, y,\, z). \tag{1a}$$

It *manifestly* leaves *time* invariant, i.e., $t' = t$. To grasp the "inversion by boost reversal" property which is *incorporated* into the Galilean transformation of Eq. (1a), we must *invert* it, which clearly produces,

$$(t,\, x,\, y,\, z) = (t',\, (x' + vt')\,,\, y',\, z')\,. \tag{1b}$$

We can now see that this *inverse* of the Eq. (1a) Galilean transformation is *also* a Galilean transformation, one which transforms the inertial frame of reference *by the addition of the x-direction constant-velocity boost* $(-v, 0, 0)$, which is equal in magnitude and opposite in direction to the original boost velocity $(v, 0, 0)$. Therefore, *if* the "inversion by boost reversal" property *holds*, an observer B, *who himself is at rest in a frame of reference that is moving at velocity* $(v, 0, 0)$ *relative to observer* A, can make *the transformation of relative velocity* $(-v, 0, 0)$ from his own (primed) space-time coordinates in order to understand what observer A perceives in *his* (unprimed) space-time coordinates, while observer A of course can make *the transformation of relative velocity* $(v, 0, 0)$ from his own (unprimed) space-time coordinates in order to understand what observer B perceives in *his* (primed) space-time coordinates.

Special relativity and Galilean relativity have the "inversion by boost reversal" property *in common with each other*, but of course *they part company with each other* in that Galilean transformations *leave time invariant* (i.e., $t' = t$), whereas Lorentz transformations *leave the speed of light invariant*. The concept of leaving time invariant under a transformation of the inertial frame of reference is expressed *directly in terms of the space-time coordinates* as simply $t' = t$, while the concept of leaving the speed of light invariant under a transformation of the inertial frame of reference *involves the magnitude of velocities instead of involving the space and time coordinates directly*. Therefore it is firstly necessary to deal with transformations of *velocity* instead of the Eqs. (1a) and (1b) types of transformations of the space-time coordinates $(t,\, x,\, y,\, z)$. But that by itself isn't sufficient, since one intends to deal with the transformation of *the magnitude of velocity* rather than with the transformation of *the components of velocity*.

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Equalities involving the transformation of velocity components are pared down into equalities which predominantly involve the untransformed and transformed velocity magnitudes—to the extent feasible. Consequences of demanding light-speed invariance are produced by substituting the constant c for each velocity magnitude which occurs; the resulting equations can be solved for information of interest.

Turning our attention now to less technical, more readily visualized consequences of light-speed invariance, a salient one is the increase in the "tick" time interval of the conceptual light clock [1, 2] when that kind of clock is in motion at constant velocity $\mathbf{v}$. The effect is due to invariance of light-speed in conjunction with the patently longer path which the clock's light traverses between "ticks" when the clock is in motion; the light path is lengthened by the celebrated special-relativistic time dilation factor $(1-(|\mathbf{v}|/c)^2)^{-\frac{1}{2}}$. Furthermore, if two observers A and B who travel at relative constant velocity $\mathbf{v}$ *each* have a light clock, it is *obvious* that *each one* will perceive the *other's* light clock "tick" more slowly *by precisely this factor* than his *own* light clock which is *stationary* with respect to him. Thus it is *obvious* from the consideration of light clocks that special-relativistic time dilation *is reciprocal between inertial frames*; this aspect of *the "inversion by boost reversal" property of inertial frames*, namely that one simply takes $\mathbf{v} \to -\mathbf{v}$ *in order to switch between those frames* is part and parcel *of the impact of light-speed invariance on the special-relativistic transformation of time between inertial frames*. In other words, the imposition of light-speed invariance between all inertial frames *seems to make it outright unnecessary to separately impose the "inversion by boost reversal" property* between inertial frames.

In fact, the implications of the light clock seem almost synonymous with those of the Lorentz transformation. In particular, it is very well known that special-relativistic time dilation is readily combined with "time of flight" considerations to deduce special-relativistic *length contraction* [3]. The "other side of the coin" relation of special-relativistic length contraction to special-relativistic time-dilation is made patent by reflection on the range through the atmosphere before their decay of cosmic-ray secondary muons which are created in the upper atmosphere with large values of $(1-(|\mathbf{v}|/c)^2)^{-\frac{1}{2}}$. Although the speed of such muons is a smidgen less than c, they can have a range through the atmosphere much greater than c times their natural lifetime at rest because of special-relativistic time dilation. From the standpoint of the muon, however, *there is no time dilation at all of its natural "at rest" lifetime*; its enhanced range through the atmosphere before decay is understood by special-relativistic length contraction of the atmosphere [3] by the factor $(1-(|\mathbf{v}|/c)^2)^{\frac{1}{2}}$. Since special-relativistic length contraction can be regarded as merely an aspect of special-relativistic time dilation, it is obviously *just as reciprocal between inertial frames as special-relativistic time dilation is*; our observers A and B moving at relative constant velocity $\mathbf{v}$ will each perceive *the other one* to be length-contracted by the aforementioned factor. Once again we see that the imposition of light-speed invariance between all inertial frames *appears to make it simply unnecessary to separately impose the "inversion by boost reversal" property* between inertial frames. Incidentally, the reciprocity of time dilation and length contraction between inertial frames is counterintuitive *not because of its nature* but because terrestrial creatures *have had no long-term ordinary experience of it*. Distance/size reciprocity, namely *each* of two observers perceives *the other one* to decrease in size with distance, *isn't* counterintuitive simply because it has *for ages* been part of the ordinary experience of terrestrial creatures. However, at 11.2 km/s (about 40,000 km/hour) an object *could escape from the earth's gravitational influence*, yet *at that extra-terrestrial speed* special-relativistic length contraction and time dilation *are still only one part in a billion*!

We have now seen, via informal consideration of the implications of the light clock, that light-speed invariance *all by itself* accounts for special-relativistic time dilation and length contraction, and *it also accounts for the reciprocity of those effects*. Such a catalog *more or less summarizes the implications of the Lorentz transformation*, but that notwithstanding, we shall now show *that light-speed invariance imposed all by itself on the general x-direction homogeneously linear space-time counterpart of the* Eq. (1a) *Galilean transformation compels that general transformation to uniquely be the x-direction Lorentz transformation*.

The general x-direction homogeneously linear space-time transformation

The *most general* x-direction homogeneously linear space-time transformation that is nontrivial *only* for the (t, x) pair has the four-parameter form,

$$(t',\, x',\, y',\, z') = \left(\gamma_0\left(t - \left(v_0/c^2\right)x\right),\, \gamma(x - vt),\, y,\, z\right), \tag{2a}$$

where γ_0 and γ are dimensionless parameters which are independent of the value of (t, x, y, z), while v_0 and v are parameters that have the dimension of velocity and are likewise independent of the value of (t, x, y, z). The homogeneous linearity of Eq. (2a) ensures coincidence of the space-time coordinate origins, namely,

$$(t = 0,\, x = 0,\, y = 0,\, z = 0) \text{ transforms to } (t' = 0,\, x' = 0,\, y' = 0,\, z' = 0). \tag{2b}$$

The transformation of *velocity* which *corresponds* to the four-parameter general x-direction homogeneously linear transformation of *space-time* given by Eq. (2a) is,

$$(dx'/dt',\ dy'/dt',\ dz'/dt') = \frac{(dx'/dt,\ dy'/dt,\ dz'/dt)}{dt'/dt} = \frac{(\gamma((dx/dt) - v),\ dy/dt,\ dz/dt)}{\gamma_0\,(1 - (v_0/c^2)\,(dx/dt))}. \tag{3a}$$

Eq. (3a) shows that the transformation of *velocity* is in general a *rational* transformation rather than a *linear* one. In order to *ensure* that the rational *velocity* transformation given by Eq. (3a) *is well-defined*, we impose the following *two restrictions*,

$$\gamma_0 \neq 0, \tag{3b}$$

and,

$$|dx/dt| < \left(c^2/|v_0|\right). \tag{3c}$$

We now take note of *a key property* of the Eq. (3a) *transformation of velocity* (which of course *corresponds* to the Eq. (2a) *transformation of space-time*), namely,

$$(dx/dt,\ dy/dt,\ dz/dt) = (v, 0, 0) \text{ implies that } (dx'/dt',\ dy'/dt',\ dz'/dt') = (0, 0, 0). \tag{3d}$$

The result given by Eq. (3d) shows that the four-parameter general *x-direction transformation* described by Eq. (2a) or (3a) *expressly compensates for the x-direction constant velocity* $(v, 0, 0)$. Therefore, as long as,

$$|v| < \left(c^2/|v_0|\right), \tag{3e}$$

in accord with the Eq. (3c) restriction, the x-direction constant velocity $(v, 0, 0)$ ought to be identifiable as the *intrinsic* Eq. (2a) or (3a) transformation "boost" to the inertial frame of reference. There is one additional caveat, however: a *zero-velocity "boost"* to the inertial frame of reference *ought not to transform the the space-time coordinates nor the velocities at all*. Therefore the general x-direction transformation must be *the identity transformation* when its intrinsic x-direction velocity parameter v equals zero. We readily see that this reduction to *the identity transformation* when $v = 0$ will be the case for *both* the Eq. (2a) general x-direction homogeneously linear space-time transformation *and* for its Eq. (3a) velocity counterpart if and only if,

$$\gamma_0(v = 0) = 1,\ \ v_0(v = 0) = 0 \ \text{ and } \ \gamma(v = 0) = 1. \tag{4}$$

The Eq. (1a) Galilean transformation *manifestly obeys the rule set out in* Eq. (4). In fact, the Eq. (4) rule compels any physically legitimate x-direction homogeneously linear space-time transformation whose γ_0, v_0 and γ parameters *have completely fixed numerical values* to be *precisely the Galilean transformation.*

The transformation imposed by light-speed invariance

We can't manage to extract comprehensive consequences of light-speed invariance directly from Eq. (3a); we need to change it to a form which involves *the magnitudes of the untransformed and transformed velocities as predominantly as is feasible*. It *is* completely straightforward to obtain from Eq. (3a) a relation which involves only the square of the magnitude *of the transformed velocity*, namely,

$$(dx'/dt')^2 + (dy'/dt')^2 + (dz'/dt')^2 = \frac{\gamma^2((dx/dt) - v)^2 + (dy/dt)^2 + (dz/dt)^2}{(\gamma_0)^2\,(1 - (v_0/c^2)\,(dx/dt))^2}. \tag{5a}$$

The requirement of transformation-invariance of light-speed, which is to be imposed on Eq. (5a), *is stated in a precise manner as follows*,

$$\text{if } (dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 = c^2, \text{ then } (dx'/dt')^2 + (dy'/dt')^2 + (dz'/dt')^2 = c^2. \tag{5b}$$

The *imposition* of the Eq. (5b) requirement on Eq. (5a) *implies the particular consequence that*,

$$c^2\,(\gamma_0)^2\left(1 - \left(v_0/c^2\right)(dx/dt)\right)^2 = \gamma^2((dx/dt) - v)^2 + (c^2 - (dx/dt)^2), \tag{5c}$$

What we want to obtain from Eq. (5c) are *the constraints which it imposes on the three parameters γ_0, v_0 and γ that determine x-direction transformations which are of the general form set out in* Eqs. (2a) and (3a). Eq. (5c) can obviously *be restated* as the *vanishing* of a second-order polynomial in *the variable-value entity* (dx/dt). Therefore *the three coefficients* of that second order polynomial in (dx/dt) must vanish, so

Eq. (5c) produces *three equalities that involve* c^2, $(\gamma_0)^2$, v_0, γ^2 and v. The restatement of Eq. (5c) as a vanishing second-order polynomial in (dx/dt), organized into uniquely-presented powers of (dx/dt) and their coefficients is,

$$\left((\gamma_0)^2 (v_0/c)^2 - \gamma^2 + 1\right)(dx/dt)^2 - 2\left((\gamma_0)^2 v_0 - \gamma^2 v\right)(dx/dt) + \left((\gamma_0)^2 - \gamma^2 (v/c)^2 - 1\right)c^2 = 0. \tag{5d}$$

Eq. (5d) implies that $(\gamma_0)^2$ satisfies *the following three equalities*,

$$(\gamma_0)^2 = (\gamma^2 - 1)/(v_0/c)^2 = \gamma^2 (v/v_0) = \gamma^2 (v/c)^2 + 1, \tag{5e}$$

which, in turn, imply that γ^2 satisfies *the following two equalities*,

$$\gamma^2 = \left(1 - (v_0 v/c^2)\right)^{-1} = \left((v/v_0) - (v/c)^2\right)^{-1}, \tag{5f}$$

which yield the following quadratic equation for the transformation parameter v_0 in terms of v and c,

$$(v_0)^2 - v_0\left(v + (c^2/v)\right) + c^2 = 0, \tag{5g}$$

whose left-hand side is readily *factored* as follows,

$$(v_0 - v)\left(v_0 - (c^2/v)\right) = 0, \tag{5h}$$

revealing the equation's two roots,

$$v_0 = v \text{ and } v_0 = \left(c^2/v\right). \tag{5i}$$

Inserting the root $v_0 = \left(c^2/v\right)$ into Eq. (5f) makes γ^2 *equal to the undefined inverse of zero*. Therefore *the only applicable root of* Eq. (5h) *is*,

$$v_0 = v, \tag{5j}$$

which obeys the Eq. (4) rule that $v_0(v = 0) = 0$. Inserted into Eq. (3e), $v_0 = v$ yields *the restriction*,

$$|v| < c, \tag{5k}$$

and inserted into Eq. (3c) it yields $|dx/dt| < (c^2/|v|)$, which together with Eq. (5k) implies *the restriction*,

$$|dx/dt| \leq c. \tag{5l}$$

Inserted into Eq. (5f), $v_0 = v$ yields,

$$\gamma^2 = \left(1 - (v/c)^2\right)^{-1}, \tag{5m}$$

for which the Eq. (5k) *restriction is needed to ensure that the parameter* γ *is a real-valued finite number*. Inserting Eq. (5m) and $v_0 = v$ into Eq. (5e) yields,

$$(\gamma_0)^2 = \left(1 - (v/c)^2\right)^{-1}. \tag{5n}$$

Although Eq. (5n) is compatible with,

$$\gamma_0 = \pm\left(1 - (v/c)^2\right)^{-\frac{1}{2}}, \tag{5o}$$

the Eq. (4) rule that $\gamma_0(v = 0) = 1$, *selects* $\pm = +$ in Eq. (5o). Likewise, although Eq. (5m) is compatible with,

$$\gamma = \pm\left(1 - (v/c)^2\right)^{-\frac{1}{2}}, \tag{5p}$$

Eq. (4), which requires that $\gamma(v = 0) = 1$, *selects* $\pm = +$ in Eq. (5p).

We thus see that the transformation-invariance of light-speed, together with the Eq. (4) rule that an inertial-frame constant-velocity "boost" transformation must be the identity transformation when that "boost" velocity vanishes altogether, yields the three *completely-determined transformation parameter values*,

$$\gamma_0 = \left(1 - (v/c)^2\right)^{-\frac{1}{2}}, \;\; v_0 = v \;\; \text{and} \;\; \gamma = \left(1 - (v/c)^2\right)^{-\frac{1}{2}}, \tag{6a}$$

which, on insertion into the Eq. (2a) *general* x-direction homogeneously linear space-time transformation yield,

$$(t',\, x',\, y',\, z') = \left(\gamma\left(t - \left(v/c^2\right)x\right),\, \gamma(x - vt),\, y,\, z\right) \quad \text{where} \quad \gamma = \left(1 - (v/c)^2\right)^{-\frac{1}{2}}. \tag{6b}$$

Eq. (6b) *is precisely the x-direction space-time Lorentz transformation* [4]. Its *inverse* transformation is readily verified to be,

$$(t,\, x,\, y,\, z) = \left(\gamma\left(t' + \left(v/c^2\right)x'\right),\, \gamma(x' + vt'),\, y',\, z'\right), \tag{6c}$$

which *differs* from the *direct* Lorentz transformation of Eq. (6b) *only* in that $v \to -v$. Thus the Lorentz transformation *indeed conforms with the "inversion by boost reversal" property that was pointed out below* Eq. (1b) *in connection with the Galilean transformation*.

The above derivation of the Lorentz transformation *manifestly doesn't assume that "inversion by boost reversal" holds*; it assumes *only* the Eq. (5b) transformation-invariance of light-speed, which it *imposes* on the Eq. (5a) specialized consequence of the Eq. (3a) *general x-direction velocity transformation*.

An important characteristic of the Eq. (6b) special-relativistic Lorentz *space-time transformation* is that *it preserves the Minkowski quadratic form* $(ct)^2 - x^2 - y^2 - z^2$, namely,

$$\begin{aligned}(ct')^2 - (x')^2 - (y')^2 - (z')^2 = \gamma^2\left[(ct - (v/c)x)^2 - (x - vt)^2\right] - y^2 - z^2 = \\ \gamma^2\left[(ct)^2\left(1 - (v/c)^2\right) - x^2\left(1 - (v/c)^2\right)\right] - y^2 - z^2 = \\ \gamma^2\left[(ct)^2\gamma^{-2} - x^2\gamma^{-2}\right] - y^2 - z^2 = (ct)^2 - x^2 - y^2 - z^2.\end{aligned} \tag{7a}$$

Setting the Minkowski quadratic form $(ct)^2 - x^2 - y^2 - z^2$ to *zero* of course describes the space-time locus of the spherical shell of light, centered on the origin ($x = 0$, $y = 0$, $z = 0$), whose radius expands at the speed c, starting at the radius-value zero at time $t = 0$. That expanding spherical light shell is the consequence of releasing a light pulse of infinitesimal duration and extent at the space-time point ($t = 0$, $x = 0$, $y = 0$, $z = 0$); its space-time locus $(ct)^2 - x^2 - y^2 - z^2 = 0$ is called the light cone. The transformation-invariance of light-speed implies, inter alia, that,

$$\text{if } (ct)^2 - x^2 - y^2 - z^2 = 0, \text{ then } (ct')^2 - (x')^2 - (y')^2 - (z')^2 = 0, \tag{7b}$$

namely the transformation-invariance of the light cone. The Eq. (7a) *transformation-invariance of the Minkowski quadratic form* is of course a *stronger* condition than is the Eq. (7b) transformation-invariance *of only the light-cone space-time locus*.

An attempt to *obtain* the Lorentz transformation from imposition of the Eq. (7b) transformation-invariance of the light cone on Eq. (2a) will come up short. An *additional* imposition on Eq. (2a) is needed; it *traditionally* is the imposition on Eq. (2a) of the "inversion by boost reversal" property.

Inversion by boost reversal + light-cone invariance = Lorentz transformation

For amusement, let's derive the Lorentz transformation *again*, this time *following a far more traditional path*. First we impose the "inversion by boost reversal" property on the Eq. (2a) general x-direction homogeneously linear space-time transformation, constrained by Eq. (4), and then we *combine* the *consequence thereof* with Eq. (2a) *and* the imposition of the Eq. (7b) transformation-invariance of the light cone.

To impose the "inversion by boost reversal" property on the Eq. (2a) general x-direction homogeneously linear space-time transformation, we must first *invert* that transformation, with the result,

$$(t,\, x,\, y,\, z) = \left(\frac{\left(t' + (\gamma_0/\gamma)\left(v_0/c^2\right)x'\right)}{\left(\gamma_0\left(1 - \left(vv_0/c^2\right)\right)\right)},\, \frac{\left(x' + (\gamma/\gamma_0)\, vt'\right)}{\left(\gamma\left(1 - \left(vv_0/c^2\right)\right)\right)},\, y',\, z'\right). \tag{8a}$$

For the Eq. (8a) *inversion* of Eq. (2a) to be accomplished *by the sign reversal* $v \to -v$, it is *necessary* that,

$$\gamma_0 = \gamma = \pm\left(1 - \left(vv_0/c^2\right)\right)^{-\frac{1}{2}}, \text{ and } \textit{also} \text{ that } v_0 \textit{ be an odd function of } v. \tag{8b}$$

The $\pm$ sign ambiguity which appears in Eq. (8b) for γ and γ_0 *must be resolved in favor of* $\pm = +$ in order to satisfy the Eq. (4) rule that $\gamma_0(v = 0) = 1 = \gamma(v = 0)$. Note that the Eq. (8b) upshot of imposing the "inversion by boost reversal" property *is fully compatible with the Galilean transformation*: with $\pm = +$, the parameter choice $v_0 = 0$, which makes v_0 a (trivial) *odd function of* v, causes Eq. (8b) *to in addition yield* *the two parameter values* $\gamma_0 = \gamma = 1$*, which completes the entire parameter-value description of the Galilean transformation.*

Now use Eq. (2a) and $\gamma_0 = \gamma$ from Eq. (8b) to evaluate ct', x', y' and z' in Eq. (7b). After that is done, Eq. (7b) becomes,

$$\text{if } (ct)^2 - x^2 - y^2 - z^2 = 0, \text{ then } \gamma^2 \left((ct - (v_0/c)x)^2 - (x - vt)^2 \right) - y^2 - z^2 = 0. \tag{8c}$$

Now specialize Eq. (8c) to the case that $x \neq 0$ and $y = z = 0$. The upshot of these assumptions is for Eq. (8c) to become,

$$\text{if } t = \pm x/c, \text{ then } \gamma^2 \left(2tx(v - v_0) + (v_0/c)^2 x^2 - v^2 t^2 \right) = 0, \tag{8d}$$

whose consequence is the two equations,

$$\gamma^2 x^2 \left(\pm 2((v/c) - (v_0/c)) + (v_0/c)^2 - (v/c)^2 \right) = ((v_0/c) - (v/c))\gamma^2 x^2 (\mp 2 + (v_0/c) + (v/c)) = 0, \tag{8e}$$

which are *both* satisfied by,

$$v_0 = v. \tag{8f}$$

With the sign $\pm = +$ in Eq. (8b), Eqs. (8f) and (8b) imply that,

$$\gamma_0 = \gamma = \left(1 - (v/c)^2 \right)^{-\frac{1}{2}}. \tag{8g}$$

The results given by Eqs. (8f) and (8g) are exactly the same as those given by Eq. (6a), namely the three parameters of the Lorentz transformation. Thus this far more traditional path of imposition of the "inversion by boost reversal" property and the Eq. (7b) transformation-invariance of the light cone yields the same Lorentz-transformation result as the path of *imposition of comprehensive transformation-invariance of light-speed* that is described in detail by Eqs. (2a), (3a), (5a) and (5b), with Eq. (5c) *being its consequence*.

References

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