Question 433:Some Integration Questions

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abstract

This note presents some integrals involving pi.

1. INTRODUCTION. The constant pi is defined by

$$\pi = 4\int_0^1 \frac{1}{1+x^2}\,dx = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265... \tag{1}$$

2. A DEFINITE INTEGRAL FOR PI.

$$1-\frac{\pi}{4} = \int_0^1 x\sqrt{\frac{1-x}{1+x}}\,dx \tag{2}$$

For $v = \left(\frac{\sqrt{5}-1}{2}\right)\sqrt{\sqrt{5}-2}$ we have

$$1-\frac{\pi}{4} = \\ = \frac{2}{3}\int_0^v \sqrt{1-3x^2}\left(\cos\left(\frac{4\pi}{3}+\frac{1}{3}\cos^{-1}\frac{18x^2-1}{\left(1-3x^2\right)^{3/2}}\right) - \cos\left(\frac{2\pi}{3}+\frac{1}{3}\cos^{-1}\frac{18x^2-1}{\left(1-3x^2\right)^{3/2}}\right)\right)dx \tag{3}$$

$$1-\frac{\pi}{4} = \frac{2}{\sqrt{3}}\int_0^v \sqrt{1-3x^2}\,\sin\left(\frac{1}{3}\cos^{-1}\frac{18x^2-1}{\left(1-3x^2\right)^{3/2}}\right)dx \tag{4}$$

$$\int_0^v \sqrt{1-3x^2}\,\cos\left(\frac{2\pi}{3}+\frac{1}{3}\cos^{-1}\frac{18x^2-1}{\left(1-3x^2\right)^{3/2}}\right)dx = \frac{\left(2\sqrt{5}-5\right)\sqrt{\sqrt{5}-2}}{2} \\ -\frac{3}{\sqrt{2}}\left(\frac{1}{3}\left(\frac{3-\sqrt{5}}{2}\right)^{3/2} F\left(\frac{1}{2},\frac{3}{2};\frac{5}{2};\frac{3-\sqrt{5}}{2}\right) - \frac{1}{5}\left(\frac{3-\sqrt{5}}{2}\right)^{5/2} F\left(\frac{1}{2},\frac{5}{2};\frac{7}{2};\frac{3-\sqrt{5}}{2}\right)\right) \tag{5}$$

$$\int_0^{v}\sqrt{1-3x^2}\cos\left(\frac{4\pi}{3}+\frac{1}{3}\cos^{-1}\frac{18x^2-1}{\left(1-3x^2\right)^{3/2}}\right)dx=\frac{3}{2}-\frac{3\pi}{8}+\frac{\left(2\sqrt{5}-5\right)\sqrt{\sqrt{5}-2}}{2}$$
$$-\frac{3}{\sqrt{2}}\left(\frac{1}{3}\left(\frac{3-\sqrt{5}}{2}\right)^{3/2}F\left(\frac{1}{2},\frac{3}{2};\frac{5}{2};\frac{3-\sqrt{5}}{2}\right)-\frac{1}{5}\left(\frac{3-\sqrt{5}}{2}\right)^{5/2}F\left(\frac{1}{2},\frac{5}{2};\frac{7}{2};\frac{3-\sqrt{5}}{2}\right)\right) \tag{6}$$

Remark : $F(a,b;c;x)$ is the hypergeometric function.

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