Ramanujan's Roots

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abstract

This note presents some formulas related with pi

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1. INTRODUCTION. Ramanujan established many intriguing equalities between radicals, [60],[61],[62],[63]. In particular:

$$\sqrt[3]{\sqrt[3]{2}-1}=\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}} \tag{1}$$

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}}=\frac{1}{3}\left(\sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25}\right) \tag{2}$$

$$\sqrt[6]{7\sqrt[3]{20}-19}=\sqrt[3]{\frac{5}{3}}-\sqrt[3]{\frac{2}{3}} \tag{3}$$

$$\sqrt[3]{\sqrt[5]{\frac{32}{5}}-\sqrt[5]{\frac{27}{5}}}=\sqrt[5]{\frac{1}{25}}+\sqrt[5]{\frac{3}{25}}-\sqrt[5]{\frac{9}{25}} \tag{4}$$

$$\sqrt[4]{\frac{3-2\sqrt[4]{5}}{3+2\sqrt[4]{5}}}=\frac{\sqrt[4]{5}-1}{\sqrt[4]{5}+1} \tag{5}$$

This note presents some formulas related with the radical (5).

Notation and some relations:

$$r=\sqrt[4]{\frac{3-2\sqrt[4]{5}}{3+2\sqrt[4]{5}}}=\frac{\sqrt[4]{5}-1}{\sqrt[4]{5}+1}=\frac{3+\sqrt{5}-\sqrt[4]{5}-\sqrt[4]{125}}{2} \tag{6}$$

$$r=\sqrt[4]{161+72\sqrt{5}-108\sqrt[4]{5}-48\sqrt[4]{125}} \tag{7}$$

$$r=\frac{3+\sqrt{5}-\sqrt{10+6\sqrt{5}}}{2} \tag{8}$$

$$r=\frac{1}{\sqrt{6+3\sqrt{5}+2\sqrt{20+9\sqrt{5}}}} \tag{9}$$

$$r=\cfrac{1}{3+\sqrt{5}-\cfrac{1}{3+\sqrt{5}-\cfrac{1}{3+\sqrt{5}-\dots}}} \tag{10}$$

$$\frac{1}{r} = 2 + \sqrt[3]{9 - \sqrt{5} + 7\sqrt[3]{9 - \sqrt{5} + 7\sqrt[3]{9 - \sqrt{5} + \ldots}}} \tag{11}$$

$$r^3 - 6r^2 + 5r = 8\cos\left(\frac{2\pi}{15}\right)\sin\left(\frac{\pi}{30}\right) \tag{12}$$

$$r = 2 - 2\sqrt{\frac{7}{3}}\cos\left(\frac{\pi}{3} - \frac{1}{3}\cos^{-1}\left(\frac{\sqrt{21}\left(27 - 3\sqrt{5}\right)}{98}\right)\right) \tag{13}$$

$$r = 2 - 2\sqrt{\frac{7}{3}}\cos\left(\frac{\pi}{6} + \frac{1}{3}\sin^{-1}\left(\frac{\sqrt{21}\left(27 - 3\sqrt{5}\right)}{98}\right)\right) \tag{14}$$

$$r = 2 - 2\sqrt{\frac{7}{3}}\sin\left(\frac{\pi}{6} + \frac{1}{3}\cos^{-1}\left(\frac{\sqrt{21}\left(27 - 3\sqrt{5}\right)}{98}\right)\right) \tag{15}$$

$$r = 2 - 2\sqrt{\frac{7}{3}}\sin\left(\frac{\pi}{3} - \frac{1}{3}\sin^{-1}\left(\frac{\sqrt{21}\left(27 - 3\sqrt{5}\right)}{98}\right)\right) \tag{16}$$

$$r = \sqrt{-1 + 2\left(\cos\left(\tan^{-1}\frac{1}{\sqrt[4]{5}}\right) + \sin\left(\tan^{-1}\frac{1}{\sqrt[4]{5}}\right)\right)^{-2}} \tag{17}$$

$$r = \tan\left(\frac{\pi}{4} - \tan^{-1}\frac{1}{\sqrt[4]{5}}\right) \tag{18}$$

2. SOME FORMULAS FOR PI.

$$\pi = 3\sin^{-1}\left(\frac{6 - 3r}{2\sqrt{21}}\right) + \sin^{-1}\left(\frac{\sqrt{21}\left(27 - 3\sqrt{5}\right)}{98}\right) \tag{19}$$

$$\pi = 2\sin^{-1}\left(\sqrt{\frac{3r}{1 + r^2}}\right) + 2\sin^{-1}\left(\sqrt{\frac{r\sqrt{5}}{1 + r^2}}\right) \tag{20}$$

$$\pi = 2\sin^{-1}\left(\sqrt{2r + 5r^4}\right) + 2\sin^{-1}\left(\sqrt{3r - r^5}\right) \tag{21}$$

$$\pi = 5\sqrt{\frac{3-\sqrt{5}}{2}} \sum_{n=0}^{\infty} \left(\frac{5r}{32}\right)^n f_n \tag{22}$$

$$f_n = \sum_{k=0}^{n} \left(-\frac{192}{25}\right)^k \sum_{m=0}^{n-k} \left(-\frac{16}{15}\right)^m \binom{k}{m} \binom{n-k-m}{k} \binom{2n-2k-2m}{n-k-m} \frac{1}{2n-2k-2m+1} \tag{23}$$

$$\pi = 4\tan^{-1}(r) + 4\tan^{-1}\left(\frac{1}{\sqrt[4]{5}}\right) \tag{24}$$

$$\pi = 4\tan^{-1}(r) + 4\sin^{-1}\left(\frac{1}{\sqrt{1+\sqrt{5}}}\right) \tag{25}$$

$$\pi = 4\sin^{-1}\left(\frac{r}{\sqrt{1+r^2}}\right) + 4\tan^{-1}\left(\frac{1}{\sqrt[4]{5}}\right) \tag{26}$$

$$\pi = 4\tan^{-1}(r) + \frac{4}{\sqrt[4]{5}} F\left(\frac{1}{4},1;\frac{5}{4};\frac{1}{5}\right) - \frac{4}{\sqrt[4]{125}} F\left(\frac{3}{4},1;\frac{7}{4};\frac{1}{5}\right) \tag{27}$$

Remark: F is the hypergeometric function.

$$\pi = \frac{5}{2}\sqrt{\frac{5-\sqrt{5}}{2}} \sum_{n=0}^{\infty} r^n g_n \tag{28}$$

$$g_n = \sum_{m=0}^{[2n/3]} \sum_{k=0}^{n-m} \frac{2^{-3k}}{2k+1} \binom{2k}{k} \binom{2k}{m} \binom{3k+n-m-1}{n-m-k} \tag{29}$$

$$\pi = 8\tan^{-1}(r) + 4\tan^{-1}\left(\frac{2-4r+r^2}{3}\right) \tag{30}$$

$$\pi = 2\sin^{-1}\left(r^2\sqrt{5}\right) + 2\sin^{-1}\left(\sqrt{6r\left(1-r^2\right)}\right) \tag{31}$$

$$\frac{\pi}{2}\left(\frac{\sqrt[4]{5}-1}{\sqrt[4]{5}+1}\right) = \int_0^{\infty} \frac{1}{1+\left(6+3\sqrt{5}\right)x^2 + \left(20+9\sqrt{5}\right)x^4}\,dx \tag{32}$$

$$\pi = 16\tan^{-1}(r) + 16\tan^{-1}\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}} - r\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}} + r\sqrt{2-\sqrt{2+\sqrt{2}}}}\right) \tag{33}$$

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