A note on a problem involving a square in a curvilinear triangle

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Abstract. A problem involving a square in the curvilinear triangle made by two touching congruent circles and their common tangent is generalized.

Keywords. square in a curvilinear triangle

Mathematics Subject Classification (2010). 01A27, 51M04

Let \( \alpha_1 \) and \( \alpha_2 \) be touching circles of radius \( a \) with external common tangent \( t \). In this note we consider the following problem [1, 4, 5] (see Figure 1).

Problem 1. \( ABCD \) is a square such that the side \( DA \) lies on \( t \) and the points \( C \) and \( B \) lie on \( \alpha_1 \) and \( \alpha_2 \), respectively. Show that \( 2a = 5|AB| \).

![Figure 1](image1)

If \( \gamma_1, \gamma_2, \ldots, \gamma_n \) are congruent circles touching a line \( s \) from the same side such that \( \gamma_1 \) and \( \gamma_2 \) touch and \( \gamma_i \) (\( i = 3, 4, \ldots, n \)) touches \( \gamma_{i-1} \) from the side opposite to \( \gamma_1 \), then \( \gamma_1, \gamma_2, \ldots, \gamma_n \) are called congruent circles on \( s \). The curvilinear triangle made by \( \alpha_1, \alpha_2 \) and \( t \) is denoted by \( \Delta \). The incircle of \( \Delta \) touches \( \alpha_1 \) and \( \alpha_2 \) at \( C \) and \( B \), respectively as in Figure 1. Indeed the problem is generalized as follows (see Figure 2):

Theorem 1. If \( \beta_1, \beta_2, \ldots, \beta_n \) are congruent circles on \( t \) lying in \( \Delta \) such that \( \beta_1 \) touches \( \alpha_1 \) at a point \( C \) and \( \beta_n \) touches \( \alpha_2 \) at a point \( B \) and \( A \) is the foot of perpendicular from \( B \) to \( t \), then the following relations hold.

(i) \( n|AB| = |BC| \).
(ii) \( 2a = \left( (\sqrt{n} + 1)^2 + 1 \right) |AB| \).

Proof. Let \( b \) be the radius of \( \beta_1 \). By Theorem 5.1 in [2] we have

(1) \( a = (\sqrt{n} + 1)^2 b \).

Let \( d = |AB| \). Since \( C \) divides the segment joining the centers of \( \alpha_1 \) and \( \beta_1 \) in the ratio \( a : b \) internally, we have

(2) \( \frac{d - b}{b} = \frac{a - b}{a + b} \).
Eliminating $b$ from (1) and (2), and solving the resulting equation for $d$, we get
$$d = 2a/(1 + (1 + \sqrt{n})^2).$$
But in the minus sign case we get
$$2b - d = 2a(1 - 4\sqrt{n})(n^2 - n + 2\sqrt{n} + 2) < 0$$
by (1). Hence $d = 2a/(1 + (1 + \sqrt{n})^2)$. This proves (ii). Let $|BC| = 2h$. Then from the right triangle formed by the line $BC$, the segment joining the centers of $\alpha_1$ and $\beta_1$, and the perpendicular from the center of $\alpha_1$ to $BC$, we get
$$(a - h)^2 + (a - d)^2 = a^2.$$ Solving the equation for $h$, we have
$$h = a - \sqrt{(2a - d)d} = an/(1 + (1 + \sqrt{n})^2).$$ This proves (i).

The figure consisting of $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $\cdots$, $\beta_n$ and $t$ is denoted by $\mathcal{B}(n)$ and considered in [2]. The next theorem also shows that the points $B$ and $C$ lies on the incircle of $\Delta$ in Figure 1 (see Figure 3).

**Theorem 2.** Let $\beta_1$, $\beta_2$, $\cdots$, $\beta_n$ be congruent circles on a line $s$. If a circle $\alpha$ touches $s$ and $\beta_1$ and $\beta_n$ externally at points $C$ and $B$, respectively, $A$ is the foot of perpendicular from $B$ to $s$, then the following relations hold.
(i) $(n - 1)|AB| = |BC|.$
(ii) $2a = ((n - 1)^2 + 4)|AB|/4.$

Theorem 2 is proved in a similar way as Theorem 1 using the fact that the ratio of the radii of $\alpha$ and $\beta_1$ equals $(n - 1)^2 : 4$ [3]. The figure consisting of $\alpha$, $\beta_1$, $\beta_2$, $\cdots$, $\beta_n$ and $s$ is denoted by $\mathcal{A}(n)$ and considered in [2].
REFERENCES


Tohoku Univ. WDB is short for Tohoku University Wasan Material Database.