A proof of the Riemann Hypothesis

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Abstract: This paper investigates the characteristics of the power series representation of the Riemann Xi function. A detailed investigation of the behaviour of the zeros of the real part of the power series (considering the series as potentially the addition of multiple series) and the behaviour of the curve, combined with a substitution of polar coordinates in the power series and in the definition of the critical strip (leading to a critical area), and the relationship with the zeros of the imaginary part of the power series leads to the conclusion that the Riemann Xi function only has real zeros.

Introduction

This paper addresses one of the key unresolved questions arising from Riemann's original 1859 paper regarding the distribution of prime numbers ('Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse'[1] - translation in Edwards [2]) - the truth or otherwise of the Riemann Hypothesis ('One finds in fact about this many real roots within these bounds and it is very likely that all of the roots are real' - referring to the roots of the Riemann Xi function).

This paper starts from the power series representation of the Xi function ('This function is finite for all values of t and can be developed as a power series in tt which converges very rapidly')(Reimann)[1] and (Edwards)[3].

Section 1 defines the power series that is being investigated, examines the key points of the behaviour of the series coefficients and examines the shape and properties of the real part curve when there is no imaginary component.

Considering the function as the addition of a number of power series and investigating the consequences of adding power series with complex zeros leads to the conclusion that the function either has all real zeros or some real zeros and an infinite number of complex zeros (with increasing magnitude imaginary components).

Section 2 deals with the effects of a change of coordinates to polar coordinates (both on the function and on the shape of the critical strip).

Investigation of the function varying the initial constant leads to the conclusion that the function has all real zeros or some real zeros and an infinite number of complex zeros (with the magnitude of the imaginary components growing above a given size).

Section 3 investigates the paths of the zeros of both real and imaginary parts of the Xi function, showing that if the function has only real zeros when there is no imaginary component, there will be no additional entire function zeros generated when there is an imaginary component.

Section 4 develops the implications of the earlier investigations and the change of coordinates, leading to the conclusion (the proof of the Riemann Hypothesis).

1 Power Series

1.1 Original Equation

Riemann's original equation in his paper (Riemann)[1]:

ξ(t)=4$\int_1^\infty (d/dx(x^{3/2}\psi'(x)))x^{-1/4}cos(\frac{t}{2}logx)dx$

where $\psi(x) = \sum_{m=1}^{\infty} e^{-m^2\pi x}$

To avoid ξ-Ξ confusion, the equation from Edwards[3] is used:

ξ(s)=4$\int_1^\infty (d/dx(x^{3/2}\psi'(x)))x^{-1/4}cosh(\frac{1}{2}(s-\frac{1}{2})logx)dx$

This leads to (Edwards)[3]:

$\xi(s) = \sum_{n=0}^{\infty} a_{2n}(s-\frac{1}{2})^{2n}$

where a_{2n}=4$\int_1^\infty (d/dx(x^{3/2}\psi'(x))x^{-1/4}\frac{logx^{2n}}{2^{2n}(2n)!})dx$

Now, if t=(x+yi), then (s-$\frac{1}{2}$) = it = (xi-y), and:

ξ(s)=$\sum_{n=0}^{\infty} a_{2n}(xi-y)^{2n}$

Due to the fact that all a_{2n} are positive (Edwards p41)[4], it immediately follows from the above that if x=0, there are no real zeros of the function and if y=0 then there are potentially many real zeros (depending on the actual values of a_{2n}).

It is important to note at this point that it has been proven that

ξ(s)=$\sum_{n=0}^{\infty} a_{2n}(xi-y)^{2n}$

(a polynomial in tt) has been proven to converge as the coefficients decrease rapidly; this result is necessary in the convergence of the product representation (Hadamard) [5].

It is also important to note at this point that this series is a Taylor series of an analytic function (hence the partial sums - Taylor Polynomials - can be used as approximations of the function with more terms leading to improved approximations).

1.2 Coefficients

In appendix A Table 1 shows the values for the first 50 coefficients (a_{2n}) of the series. Note that they are monotonically strictly decreasing and rapidly decreasing and non-zero (necessary for rapid convergence - this will continue for all the coefficients). In addition, the coefficients are decreasing in a predictable way such that a curve drawn through the coefficients is smooth, continuously decreasing and non-oscillatory. As |(xi-y)|increases, then the number of terms in the series expansion needed for accuracy of the result increases.

Figures 1 and 2 illustrate this.

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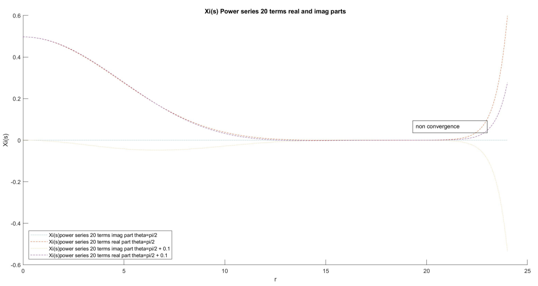

Fig. 1: $\xi(s)$ First 20 terms in power series.

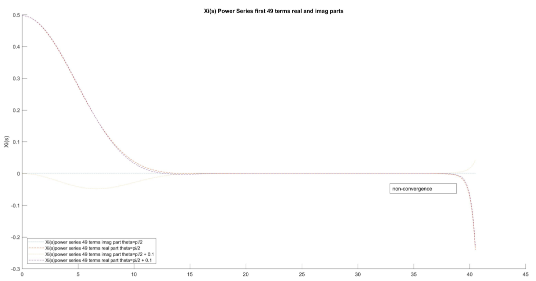

Fig. 2: $\xi(s)$ First 49 terms in power series. Note greater r before non convergence.

1.3 Real Curve Shape Components

Secondly investigating some of the properties of the terms of the Riemann definition of ξ(s) (Edwards P16)[6] for $s = (\frac{1}{2} + ri)$; this is the equivalent of varying x and setting y=0 in the power series (we will see below in the polar coordinates section why r is used as the variable):

ξ(s) = $\Pi(\frac{s}{2})(s-1)\pi^{\frac{-s}{2}}\zeta(s)$

Since we know that this expression has the same zeros as $\zeta(s)$ and no other zeros, we can consider the magnitudes of the various components.

a) The $\Pi(\frac{s}{2})$ term, where Π is the factorial function. The magnitude of the factorial function for real numbers of increasing size increases rapidly.

However, the behaviour for complex numbers with a fixed real part and an imaginary part of increasing size is very different - the magnitude of the function decreases very rapidly with increasing imaginary number size.

In addition, it is oscillatory for both real and complex components.

This behaviour can be seen by investigating the product representation of the factorial function for the complex number s as known to Euler [8]:

$$\Pi(s) = \lim_{N\to\infty} \frac{N!}{(s+1)(s+2)...(s+N)}(N+1)^s$$

For s=(a+bi), where |a|<1, then, for any N, as b increases in magnitude both real and imaginary components of the $(N+1)^s$ term are oscillatory with a constant magnitude of oscillation, while the denominator increases rapidly in magnitude.
In the limit, this leads to a function of rapidly decreasing magnitude.

See Figure 3 and Figure 4 for illustrations.

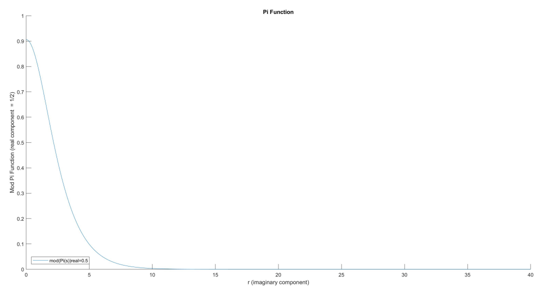

Fig. 3: Mod $\Pi(\frac{1}{2} + ri)$, r$<$ 18

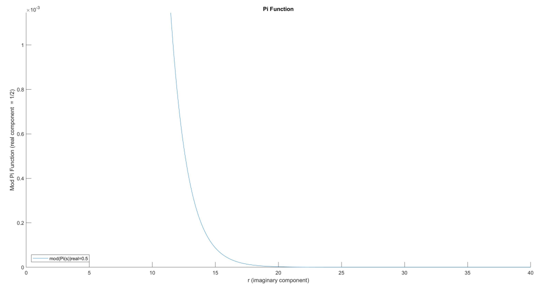

Fig. 4: Mod $\Pi(\frac{1}{2} + ri)$, r$<$ 30 Note rapid decrease in magnitude

b) (s-1) The magnitude of this (non oscillatory) term increases slowly with the magnitude of s.

c) $\pi^{\frac{-s}{2}}$. The real and imaginary components of this term oscillate with a fixed magnitude of oscillation. The magnitude of the term is constant for fixed real part value and varying imaginary part.

d) $\zeta(s)$. The real and imaginary components of this term oscillate with a very slowly increasing magnitude (see Figure 5 for illustration).

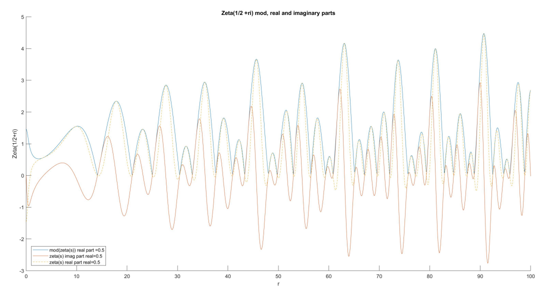

Fig. 5: $\zeta(\frac{1}{2} + ri)$

Looking at the product of the individual terms, this means that the curve is oscillatory with decreasing magnitude of oscillation as r increases.

In addition, since we know that for b=0 then there is no imaginary element of the function, the function tends to zero in the limit (as expected).

This reduction in magnitude can be seen in the curves with the actual a_{2n} below. Figure 6 and Figure 7 show 2 sections of ξ(s) with b=0.

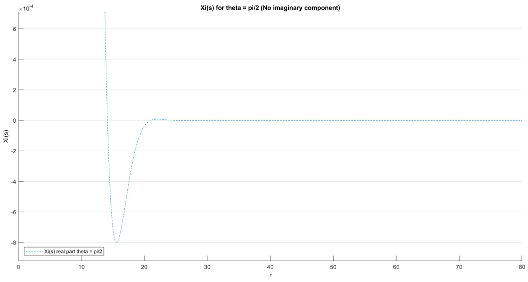

Fig. 6: $\xi(s)$ No imaginary component, r< 25

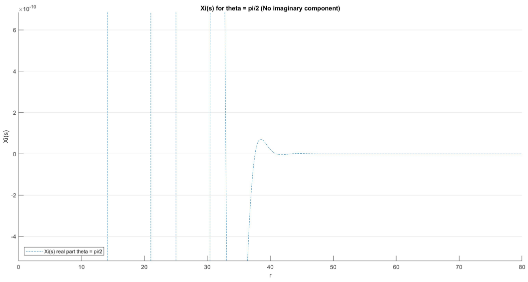

Fig. 7: $\xi(s)$ No imaginary component, r< 40 Note rapid decrease in magnitude

1.4 Real Curve Characteristics

Looking at ξ(s)=$\sum_{n=0}^{\infty} a_{2n}(xi-y)^{2n}$ in more detail:

Firstly setting y=0 to give ξ(s)=$\sum_{n=0}^{\infty} a_{2n}(xi)^{2n}$

Expanding ξ(s)=$\sum_{n=0}^{\infty} a_{2n}(xi)^{2n}$=$a_0 - a_2x^2 + a_4x^4 - a_6x^6 + a_8x^8$...

This is a Taylor series consisting of a positive constant and an infinite number of even powers of x with alternating signs for each term of the series. As noted above, the coefficients are rapidly decreasing and non-zero. Also as noted above, using a reduced number of terms in the series results in a rapidly increasing (or decreasing) value of the function for x larger than a certain value.

At this point, we know that the first M (M not known, but large) zeros are real, the magnitude of the oscillation of the curve tends to reduce as x increases and that there are no complex zeros with the imaginary component greater than $\frac{1}{2}$.

Investigating the possible behaviour of the curve:

In general terms, a series of this type (constant plus even powers of x with alternating signs) will produce a curve that is oscillatory about another curve. With appropriate coefficients, then the curve will be oscillatory about the horizontal axis. If the curve is oscillatory about the horizontal axis, then it is possible for the function to have only real zeros.

The curve can have either all real zeros or might not have all real zeros.

In general terms, a series of this type can be considered to be the sum of a number of power series.

If it does not have all real zeros, then it can be considered to be the addition of a function with all real zeros and a function (or functions) with real and/or complex zeros.

Looking in more detail at possible functions with (not necessarily exclusively) complex zeros (having components consisting of a constant and powers of x^{2n}):

Firstly, if one of the functions with complex zeros is a polynomial, then for large enough x, the curve will have a rapidly increasing (or decreasing) value, leading to complex zeros with imaginary components of increasing magnitude (at some point with the imaginary component being larger than $\frac{1}{2}$.

Secondly, the functions with complex zeros could be power series, consisting of a constant and an infinite number of terms of powers of x^{2n} (not necessarily including all of the powers of x^{2n}). Investigating first the properties of a power series with only terms of powers of x^{2n} (leaving the constant for later consideration):

First we assume that the combined function has at some point one pair of complex zeros immediately followed by a pair of real zeros. This can be generated at minimum by the addition of 2 terms to the function with all real zeros:

$b_{2n}x^{2n}$-$b_{2n+4}x^{2n+4}$ (or with the signs reversed).

Note that this format is necessary for generating the sequence real zero pair, complex zero pair, real zero pair in a function consisting of powers of x^{2n}.

Note that the coefficients b_{2n} in this case will be smaller than a_{2n}.

If only these two terms are added, then a polynomial has been added.

However, a more interesting possibility is that these two terms are the first two in a power series of the form:

$b_{2n}x^{2n} - b_{2n+4}x^{2n+4} + b_{2n+8}x^{2n+8} - b_{2n+12}x^{2n+12}$+...

Rearranging:

= $x^{2n}(b_{2n} - b_{2n+4}x^4 + b_{2n+8}x^8 - b_{2n+12}x^{12}$+...)

Note that this series has roots of the form $(x^4 - \alpha)(x^4 - \beta)(x^4 - \gamma)$... = 0,

meaning that there are infinite complex roots, with the imaginary components of the complex roots increasing as x increases (and at some point being larger than $\frac{1}{2}$).

The same argument is valid for sequences with any finite number of pairs (say q) of complex zeros between pairs of real zeros:

Power series: $x^{2n}(b_{2n} - b_{2n+(1+q)}x^{(1+q)} + b_{2n+2(1+q)}x^{2(1+q)} - b_{2n+3(1+q)}x^{3(1+q)}$+...).

Roots of the form: $(x^{(1+q)} - \alpha)(x^{(1+q)} - \beta)(x^{(1+q)} - \gamma)$... = 0,

Also, with the substitution of y=$x^{(1+q)}$, this can also be written as:

$y^{\frac{2n}{(1+q)}}(b_{2n} - b_{2n+(1+q)}y + b_{2n+2(1+q)}y^2 - b_{2n+3(1+q)}y^3$...).

At this point, it is important to note the decreasing magnitude of the oscillations of the ξ function. If an added function with complex zeros has a magnitude large enough at some point to generate complex zeros for the combined function, then the magnitude of oscillation of the function with complex zeros must be either: increasing (leading to infinite complex zeros after the first pair of complex zeros), constant (leading to infinite complex zeros after the first pair of complex zeros) or decreasing more slowly than the ξ function (leading to infinite complex zeros after the first pair of complex zeros). In each case, there would be infinite complex zeros with increasing imaginary components, at some point greater than $\frac{1}{2}$.

It is important to note that the combined function could be the sum of any number of functions - the result above shows that if any of the functions generate complex zeros, then it will generate infinite complex zeros with imaginary components at some point being greater than $\frac{1}{2}$. We know that the ξ function does not have complex zeros with imaginary component greater than $\frac{1}{2}$, so we know that the x^{2n} coefficients are those of a function with only real zeros.

Another point to consider is the nature of the coefficients of the expression:

$a_0 - a_2x^2 + a_4x^4 - a_6x^6 + a_8x^8...$

As noted above in section 1.2, drawing a curve through these coefficients produces a non-oscillatory graph.

However, if the combined function included a power series of the form:

$x^{2n}(b_{2n} - b_{2n+(1+q)}x^{(1+q)} + b_{2n+2(1+q)}x^{2(1+q)} - b_{2n+3(1+q)}x^{3(1-}$

then the graph of the coefficients would include an oscillating element due to the $b_{2n+(m+q)}$ components. The curve of the coefficients of the power series of the ξ function does not have an oscillating element, leading to the conclusion that the function does not include a power series of the form:

$x^{2n}(b_{2n} - b_{2n+(1+q)}x^{(1+q)} + b_{2n+2(1+q)}x^{2(1+q)} - b_{2n+3(1+q)}x^{3(1-}$ $+...)$

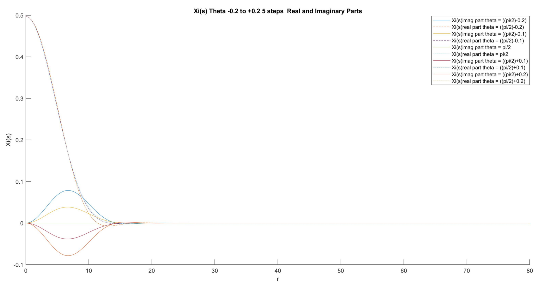

Fig. 8: $\xi(s)$ with varying θ

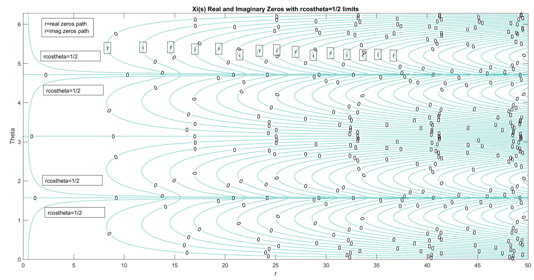

Fig. 9: $\xi(s)$ real and imaginary part zeros.

2 Polar Coordinates

2.1 Substitution

Using de Moivre's Theorem (Heading p115 [7]) $(ai - b)^{2n}$ can be rewritten as $r^{2n}(cos\theta + isin\theta)^{2n}$ and expanded as $r^{2n}cos2n\theta + ir^{2n}sin2n\theta$, taking r to range from 0 to ∞ and θ to range from $\frac{\pi}{2}$ to π for the most relevant quadrant. The structure of the expression (see below) means that the π to 2π half is a reflection of the 0 to π half. The behaviour of the expression is markedly different for $r \leq 1$. From this point, I will consider only $r > 1$ (since we know that there are no relevant zeros for $r \leq 1$).

2.2 Complete Expression

The above results in: ξ(s)=$\sum_{n=0}^{\infty} a_{2n}r^{2n}(cos\theta + isin\theta)^{2n}$

$=a_0 + a_2r^2cos2\theta + a_4r^4cos4\theta + a_6r^6cos6\theta + a_8r^8cos8\theta...$

$+i(a_2r^2sin2\theta + a_4r^4sin4\theta + a_6r^6sin6\theta + a_8r^8sin8\theta...)$

Both real and imaginary parts of the expression are single valued for each r, θ combination. Both real and imaginary parts have a period of π. For $\theta = \frac{\pi}{2}$ the expression is equal to ξ(s)=$\sum_{n=0}^{\infty} a_{2n}(ai)^{2n}$. For the actual values of a_{2n} the expression does have multiple real zeros.

Figure 8 shows the variation of the function with variation in θ.

2.3 Critical Strip to Critical Area

The polar coordinate substitution is very interesting here. The critical strip (between $\frac{1}{2} + / - \frac{1}{2}$) changes to $rcos\theta$=$+/-\frac{1}{2}$.

This means that as r increases, the width of the critical area reduces as cosθ reduces and so θ moves closer to $\theta = \frac{\pi}{2}$ leading to useful limits that can be exploited (see Figure 9).

Considering the behaviour of ξ(s)=$a_0 - a_2r^2 + a_4r^4 - a_6r^6 + a_8r^8$..., which was shown above to be an oscillating function of rapidly reducing magnitude with some real zeros, it is useful to investigate the effect of the initial constant a_0.

Since a_0 is constant, it has the effect of translating the function in the vertical direction. Remembering that the function is oscillating with a continuously decreasing magnitude and assuming that the coefficients of the powers of r are such that all real zeros are possible, then we can see that either the function can have all real zeros for one value of the constant or if the constant is varied by an amount δ (however small), then there will be a value of r above which there will be no more real zeros once δ is larger than the magnitude of the function.

Also, considering the $rcos\theta = \frac{1}{2}$ limit, then at some point (for greater r), then the distance of the imaginary zeros from the $\theta = \frac{\pi}{2}$ line will be greater than that limit (ie zeros outside the critical area) and will continue to be greater than that limit (see Figure 10).

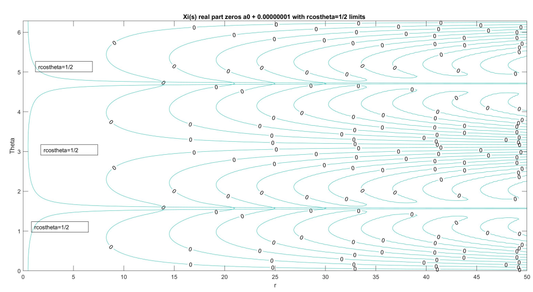

Fig. 10: $\xi(s)$ with a0 varied, real part zeros.

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3 Paths of Zeros

3.1 Real Part Zeros

Using $\theta = (\frac{\pi}{2} + \epsilon)$:

cos2$\theta = cos(2(\frac{\pi}{2} + \epsilon))$ = cosπcos2ϵ - sinπsin2ϵ = - cos2ϵ and $cos(2(\frac{\pi}{2} - \epsilon))$ = cosπcos2ϵ + sinπsin2ϵ = - cos2ϵ

Similar expressions can be generated for cos2nθ for all values of n with similar results (except alternating signs).

This means that the path of the function $a_0 + a_2r^2cos2\theta + a_4r^4cos4\theta + a_6r^6cos6\theta + a_8r^8cos8\theta$...=0 is reflected across $\theta = \frac{\pi}{2}$ for varying r.

This equation describes a family of curves.

No function in the family intersects any other function in the family (the function is single valued for each r,θ combination).

For the actual values of a_{2n}, it appears that each function in the family will pass through $\theta = \frac{\pi}{2}$.

When $\theta \neq \frac{\pi}{2}$, then we know that the function extends from the zeros on the $\theta = \frac{\pi}{2}$ line.

If a function does not pass through $\theta = \frac{\pi}{2}$, then it will have 2 reflected branches on either side of $\theta = \frac{\pi}{2}$ (non-intersecting with any other of the family of functions).

3.2 Imaginary Part Zeros

Using $\theta = (\frac{\pi}{2} + \epsilon)$:

sin2$\theta = sin(2(\frac{\pi}{2} + \epsilon))$ = sinπcos2ϵ + cosπsin2ϵ = - sin2ϵ and $sin(2(\frac{\pi}{2} - \epsilon))$ = sinπcos2ϵ - cosπsin2ϵ = + sin2ϵ

Similar expressions can be generated for sin2nθ for all values of n with similar results (except alternating signs).

This means that the path of the function $a_2r^2sin2\theta + a_4r^4sin4\theta + a_6r^6sin6\theta + a_8r^8sin8\theta$...=0 is reflected across $\theta = \frac{\pi}{2}$ for varying r.

This equation describes a family of curves.

No function in the family intersects any other function in the family (the function is single valued for each r,θ combination).

For the actual values of a_{2n} it appears that each function in the family will pass through $\theta = \frac{\pi}{2}$.

When $\theta = \frac{\pi}{2}$, then we know that the function is identically zero.

If a function does not pass through $\theta = \frac{\pi}{2}$, then it will have 2 reflected branches on either side of $\theta = \frac{\pi}{2}$ (non-intersecting with any other of the family of functions).

The real part and imaginary part have the same number of pairs of zeros (the imaginary part has an additional zero at r=0).

See Figure 9 for an illustration of the paths of real and imaginary zeros for the actual values of a_{2n} in the same graph.
Note that the paths of the zeros do not intersect in these samples (as shown above), except where the imaginary function is identically zero.

3.3 (Real + Imaginary) and (Real - Imaginary) Part Zeros

The complete function will be zero when both real and imaginary expressions are equal to each other and both zero. Thus we are looking for common zeros of these two expressions:

$a_0 + a_2r^2cos2\theta + a_4r^4cos4\theta + a_6r^6cos6\theta + a_8r^8cos8\theta$...=0 (1)
and $a_2r^2sin2\theta + a_4r^4sin4\theta + a_6r^6sin6\theta + a_8r^8sin8\theta$...=0 (2)

It is useful to investigate the combined expressions ((1)+(2)) and ((1)-(2)). If and only if they are simultaneously zero then the complete function is zero.

Reusing: cos2$\theta = cos(2(\frac{\pi}{2} + \epsilon))$ = cosπcos2ϵ - sinπsin2ϵ = - cos2ϵ and $cos(2(\frac{\pi}{2} - \epsilon))$ = cosπcos2ϵ + sinπsin2ϵ = - cos2ϵ
sin2$\theta = sin(2(\frac{\pi}{2} + \epsilon))$ = sinπcos2ϵ + cosπsin2ϵ = - sin2ϵ and $sin(2(\frac{\pi}{2} - \epsilon))$ = sinπcos2ϵ - cosπsin2ϵ = + sin2ϵ

Similar expressions can be generated for sin2nθ and cos2nθ for all values of n with similar results (except alternating signs).

(1)+(2) for ϵ is the same as (1)-(2) for - ϵ and (1)-(2) for ϵ is the same as (1)+(2) for - ϵ - that is, the expressions (1)+(2) and (1)-(2) are reflected through $\theta = \frac{\pi}{2}$ and if they cross $\theta = \frac{\pi}{2}$ then there will be a coincident pair of real zeros on $\theta = \frac{\pi}{2}$.

If they do not cross $\theta = \frac{\pi}{2}$, then there will be a reflected pair of imaginary zeros tracing reflected paths.

We know from section 1 that in fact, they do all cross $\theta = \frac{\pi}{2}$.

In addition, as the functions are single valued for each r, θ combination then there are no intersections with any other of the same family of functions.

This means that there will be no additional complete function zeros generated - each pair of imaginary part zeros will only coincide with one pair of real part zeros.

One can also see from the above expressions that as ϵ tends to $\frac{\pi}{2}$ and individual components tend to zero, the functions both tend to horizontal (i.e one would expect to see an increasing number of almost parallel, almost horizontal functions as r increases).

See Figure 11 for an illustration of the paths of the ((1)+(2)) and ((1)-(2)) zeros for the actual values of a_{2n}, Figure 12, for an illustration if we vary a_0 and Figure 13 for an illustration if we vary a_{2n}.

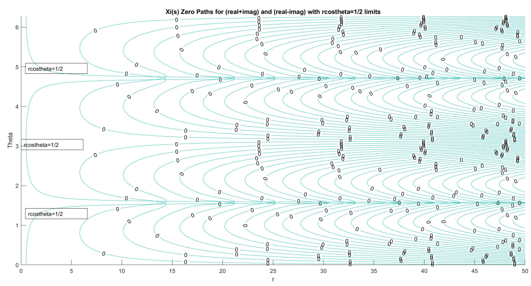

Fig. 11: $\xi(s)$ (real + imaginary) and (real - imaginary) zero paths.

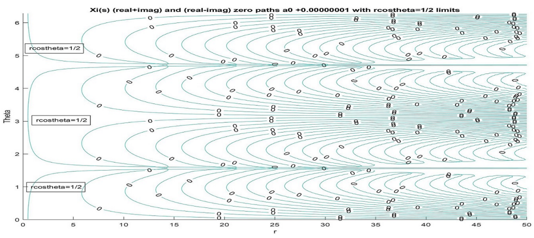

Fig. 12: $\xi(s)$ + 1E-8 (real + imaginary) and (real - imaginary) zeros.

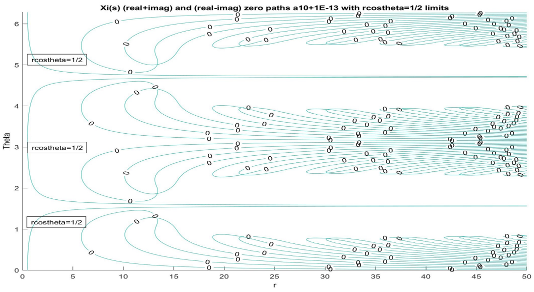

Fig. 13: $\xi(s)$ + 1E-13(r^{10}) (real + imaginary) and (real - imaginary) zeros.

4 Conclusions

Known previously - ξ(s) (with no imaginary component) does not have zeros outside the critical area/critical strip.

In Section 1.4 it was shown that, subject to the value of a_0, the real part of ξ(s) (with no imaginary component) either has all real zeros or has some real zeros and an infinite number of complex zeros with some of those zeros with an imaginary component with magnitude greater than 1/2 (ie outside the critical area/strip).

In Section 2.3 it was shown that, subject to the values of a_{2n}, the real part of ξ(s) (with no imaginary component) either has all real zeros or has some real zeros and an infinite number of complex zeros with some of those zeros with an imaginary component with magnitude greater than rcosθ = 1/2 (ie outside the critical area/strip).

In Section 3 it was shown that there are no additional zeros of the complete function due to the coincidence of imaginary zeros from the real and imaginary parts of ξ(s).

Combining these conclusions, all of the roots of the Riemann Xi function (where s=($\frac{1}{2} + ti$) - no imaginary component) are real - QED.

1 References

1 Riemann, B.: 'Gesammelte Werke.'(Teubner, Leipzig, 1892; reprinted by Dover Books, New York, 1953.) p145.
2 Edwards, H.M.: 'Riemann's Zeta Function'(Dover Publications, 2001) p299
3 Edwards, H.M.: 'Riemann's Zeta Function'(Dover Publications, 2001) p17
4 Edwards, H.M.: 'Riemann's Zeta Function'(Dover Publications, 2001) p41
5 Hadamard, J.: 'Étude sur les Propriétés des Fonctions Entières et en Particulier d'une Fonction Considérée par Riemann. *J. Math. Pures Appl.* [4] **9**, pp. 171-215 (1893)
6 Edwards, H.M.: 'Riemann's Zeta Function'(Dover Publications, 2001) p16
7 Heading, J.: 'Mathematical Methods in Science and Engineering'(2nd Edition, Edward Arnold 1970) p115
8 Euler, L.: 'De progressionibus transcendentibus....'*Comm. Acad. Sci. Petropolitanae* **5**, pp. 36-57 (1737).(Also "Opera" (1), Vol.14, pp. 1-24).

All figures created in MATLAB.

Appendix A - Coefficients

Table 1 The First 50 a_{2n} Coefficients of ξ(s) calculated numerically with MATLAB.

Coefficients	Values
a0	0.497120778
a2	0.011485972
a4	0.000123452
a6	8.32355E-07
a8	3.99223E-09
a10	1.4616E-11
a12	4.27454E-14
a14	1.03096E-16
a16	2.09977E-19
a18	3.67814E-22
a20	5.62286E-25
a22	7.59176E-28
a24	9.14334E-31
a26	9.90611E-34
a28	9.72469E-37
a30	8.7046E-40
a32	7.14349E-43
a34	5.40097E-46
a36	3.77845E-49
a38	2.45541E-52
a40	1.48738E-55
a42	8.42529E-59
a44	4.4758E-62
a46	2.23578E-65
a48	1.05272E-68
a50	4.68274E-72
a52	1.9719E-75
a54	7.87603E-79
a56	2.9891E-82
a58	1.07972E-85
a60	3.71788E-89
a62	1.22216E-92
a64	3.84066E-96
a66	1.1553E-99
a68	3.3305E-103
a70	9.2123E-107
a72	2.4475E-110
a74	6.2524E-114
a76	1.5372E-117
a78	3.641E-121
a80	8.3152E-125
a82	1.8325E-128
a84	3.9005E-132
a86	8.0242E-136
a88	1.5967E-139
a90	3.0751E-143
a92	5.7365E-147
a94	1.0371E-150
a96	1.8185E-154
a98	3.0939E-158