Analyticity and Function satisfying : $f' = e^{f^{-1}}$

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Abstract

In this note we present some new results about the analyticity of the functional-differential equation $f' = e^{f^{-1}}$ at 0 with f^{-1} is a compositional inverse of f , and the growth rate of $f_-(x)$ and $f_+(x)$ as $x \to \infty$, and we will check the analyticity of some functional equations which they were studied before and had a relashionship with the titled functional-differential and we will conclude our work with a conjecture related to Borel- summability and some interesting applications of some divergents generating function with radius of convergent equal 0 in number theory .

Keywords: Power series, Analyticity, divergent solution, Borel summability

1. Introduction

[01] Functions are used to describe natural processes and forms. By means of finite or infinite operations, we may build many types of derived functions such as the sum of two functions, the composition of two functions, the derivative function of a given function, the power series functions, etc. Yet a large number of natural processes and forms are not explicitly given by nature. Instead, they are implicitly defined by the laws of nature. Therefore we have functional equations (or more generally relations) involving our unknown functions and their derived functions. When we are given one such functional equation as a mathematical model, it is important to try to find some or all solutions, since they may be used for prediction, estimation and control, or for suggestion of alternate formulation of the original physical model. In this paper, we are interested in finding solutions that are polynomials of infinite order, or more precisely, power series functions. There are many reasons for trying to find such solutions. First of all, it is sometimes obvious from experimental observations that we are facing with

natural processes and forms that can be described by smooth functions such as power series functions. Second, power series functions are basically generated by sequences of numbers, therefore, they can easily be manipulated, either directly, or indirectly through manipulations of sequences. Indeed, finding power series solutions are not more complicated than solving recurrence relations or difference equations. Solving the latter equations may also be difficult, but in most cases, we can calculate them by means of modern digital devices equipped with numerical or symbolic packages! Third, once formal power series solutions are found, we are left with the convergence or stability problem. This is a more complicated problem which is not completely solved. Fortunately, there are now several standard techniques which have been proven useful. In this paper we join our work using some related sequences which montioned in **OEIS** which we will cite them below .Robert Anschuetz II and H. Sherwood studied in [02] this topic "When Is a Function's Inverse Equal to Its Reciprocal"? that is interesting mathematical subject dealing with multiplicative and compositional inverse in the same time , and H.Nelson proposed the functional -differential equation $f^{-1} = f'$ in [04] and it's appeared again in [05] ,And the aim of this paper is studying the behavior and analiticity of $f' = e^{f^{-1}}$ using some communs properties of the cited functional equations

2. functions satisfy :$f^{-1} = \frac{1}{f}$

Lemma 1. *let f be a function map $\mathbb{R}^*$ to itself and f^{-1} be a compositional inverse of f , one class of solution satisfies :$f^{-1} = \frac{1}{f}$*

Proof. Take any f_0 that maps $(0, 1]$ one-to-one onto $(-\infty, -1]$ with $f_0(1) = -1$. :

$$f(x) = \begin{cases} f_0(x), & \text{if } x \in (0,1] \\ \frac{1}{f_0(\frac{1}{x})}, & \text{if } x \in (1,+\infty) \\ f_0\frac{1}{x}, & x \in (-1,0) \\ \frac{1}{f_0^{-1}(x)}, & x \in (-\infty,-1] \end{cases}$$

We can look to $f_0(x) = -1 + \tan\left(\frac{(x-1)\pi}{2}\right)$ as example for that equation

3. function satisfy :$f^{-1} = f'$

As far as i know this problem was originally proposed by H. L. Nelson In [03] and appeared on page 779 in [04] it would make its way to the problem and solutions column once again in 1976 here[05] We restrict our analysis to positives real numbers because For the domain $\mathbb{R}$, no solution exists. A continuous injective $f : \mathbb{R} \to \mathbb{R}$ must be monotone, which implies that its derivative cannot change sign, but f^{-1} would include both positive and negative numbers in its range .We let that clear and obvious according to the graph shown below in figure 1 Piecing these functions together gives

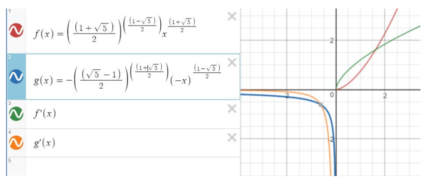

Figure 1: piecewise of f and f^{-1} show the domain of differntiability

an invertible map from $\mathbb{R}$ onto $\mathbb{R}$ such that $f'(x) = f^{-1}(x)$ when $f'(x)$ exists, and $f'(0)$ doesn't exist, but the right-hand derivative $\lim_{h \to 0+} \frac{f(h) - f(0)}{h}$ exists and equals $0 = f^{-1}(0)$. Considering that a differentiable solution is impossible, this is pretty good.

Lemma 2. *let f be a function map $\mathbb{R}_+$ onto $\mathbb{R}_+$ and f^{-1} is the compositional inverse of f ,The function satisfying the functional equation :$f'(x) = f^{-1}(x)$ is of the form :$f(x) := h(ah^{-1}(x))$ with h auxiliary function , defined in the neighborhood of $t = 0$ and coupled to f via $x = h(t)$.*

[1]

Proof. Let $a = 1 + p > 1$ be given. We shall construct a function f of the required kind with $f(a) = a$ by means of an auxiliary function h, defined in the neighborhood of $t = 0$ and coupled to f via

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$x = h(t)$, $f(x) = h(at)$, $f^{-1}(x) = h(t/a)$. The condition $f' = f^{-1}$ implies that h satisfies the functional equation

$$(01) \quad h(t/a)h'(t) = ah'(at).$$

,Writing $h(t) = a + \sum_{k\geq 1} c_k t^k$ we obtain from (01) a recursion formula for the c_k,

and one can show that $0 < c_r < 1/p^{r-1}$ for all $r \geq 1$. This means that h is in fact analytic for $|t| < p$, satisfies (01) and possesses an inverse h^{-1} in the neighborhood of $t = 0$. It

follows that the function $f(x) := h(ah^{-1}(x))$ has the required properties , it's good to show the uniqness of this solution since it's existed and well defined ,The uniqueness of the solution to the problem is established by means of the fixed point whose existence should to prove it .

4. Analyticity and Existence of fixed point for: $f^{-1} = f'$

Lemma 3. *Any solution f for the functional-differential $f^{-1} = f'$ is a real-analytic function and does have a fixed point $a \in I$*

[2]

Proof. First off, we notice that if f is a function that does the job, then f must be $\mathcal{C}^1$ and strictly increasing in $(0, \infty)$. Then, differentiating the identity

$$f(f'(x)) = x$$

repeatedly, we obtain that f is a function of class $\mathcal{C}^\infty$. What is more, we obtain that $f'' > 0$, $f''' < 0, \ldots, (-1)^k f^{(k)} > 0$; it follows from Bernstein's theorem on regularly monotonic functions as shown here in [06] that f **is a real-analytic function (see bellow footnote** on $(0, \infty)$. Now, from the identity $\frac{d}{dx} f(f(x)) = f'(f(x))f'(x) = xf'(x)$ we get that:

$$f(f(x)) = \int_0^x y\, f'(y)\, dy$$

for every $x \in I$. This allows us to ascertain that f has a fixed point $a \in I$:

[2] A function $f : \mathbb{R} \to \mathbb{R}$ is called $\mathbb{R}$-analytic iff for every $x_0 \in \mathbb{R}$ there exist $R > 0$ and power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ convergent for $|x - x_0| < R$ and such that $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ for $|x - x_0| < R$.

if this were not the case, the function $F\colon I \to \mathbb{R}$, defined for every $x \in I$ as $F(x) = f(x) - x$, would be of fixed sign. We claim that such a thing is not

possible: indeed, if $f(x) > x$ for every $x \in I$, then $y = f'(f(y)) > f'(y)$ for every $y \in (0, x)$ and whence $x < f(f(x)) = \int_0^x y\, f'(y)\, dy < \int_0^x y^2\, dy = \frac{x^3}{3}$, which doesn't necessarily hold when x is sufficiently small; since the assumption that the inequality $f(x) < x$ holds for every $x \in I$ allows us to derive a similar contradiction, we conclude that any solution f to the titled functional-differential does have a fixed point $a \in I$. Further, the strict convexity of F implies that F has at most two zeros, counting the one it has at $x = 0$. Thus, f has exactly one fixed point $a \in I$, with $f(x) < x$ in $(0, a)$, $f(x) > x$ in (a, ∞), $f'(x) > x$ in $(0, a)$, and $f'(x) < x$ in (a, ∞)

5. Uniqueness of solution for $f^{-1} = f'$

Lemma 4. *There is no other function f which satisfies all the constraints under consideration for the functional-differential $f^{-1} = f'$*

Proof. Let us suppose that f_1 and f_2 are two functions satisfying all the constraints under consideration and let $g := f_1 - f_2$. Moreover, let us denote with a_i the unique fixed point of f_i in the interval I. Without loss of generality, we can suppose that $a_1 \geq a_2$.The possibility that $a_1 > a_2$ leads us to a contradiction,Now If $a_1 = a_2 = a$, then it is not difficult to convince oneself that $0 = g(a) = g'(a) = g''(a) = \ldots$; being g a real-analytic function in I, the latter equalities implies that g vanishes identically and we are done.

Now we are ready to study the aim of this paper which include behavior of the functional equation $f' = e^{f^{-1}}$ with f map $\mathbb{R}$onto$\mathbb{R}$, before introducing our main results we try to show the preliminary analysis of the functional -differential for the derivation of some related interesting results to many area of mathematics for example : Number theory .

6. Preliminary analysis :

one might ask if there is a closed form of this equation but there is no reason to expect a closed form for it ,we can see only that there is a unique

solution in formal power series around 0 satisfying $f(0)=0$, Despite appearances, this is rather different from an ODE since the equation is non-local in the sense that the RHS at x can not be evaluated if one only knows f near x , After computations first few coefficients of the unique power series solution are [0, 1, 1/2, 0, 1/24, -1/20, 13/180, -197/1680, 2101/10080, -48203/120960, 2938057/3628800, -23059441/13305600, 74408941/19160064, -9409883317/1037836800] ,More of that the calculation of the first 100 terms of the formal power series. It is pretty clear that .It is pretty clear that $|a_n|^{-1/n} \to 0$ as $n \to \infty$ so the radius of convergence is zero, so this approach will not give a solution that is an actual function.
According to what we are cited about the preliminary analysis and observations about the titled functional equation we are ready to present the main obtained results .

7. Main results:

- (01) $f_- = f_+$ are smooth functions with $f_- = f_+$ is an actual solution to the equation $f'(x) = e^{f^{-1}(x)}$, it is merly c^∞ but not analytic having divergent power series expansion .
- (02) The equation converges in L^1 and therefore in c^∞ for $x \geq 0$.
- (03) For $h(x) = -f^{-1}(-x)$, h is totally monotonic on $[0, a[$with $0 < a \leq +\infty$ and it is invertible
- (04) h is unbounded function.
- (05) Probably Borel summation could be applied for this solution (if it is asymptotic series)
- (06) $b_n = (-1)^n n! a_n$ appears to always be a positive integer for $n > 1$ but this sequence is not in OEIS ,Also b_n it does not factorise in a way that suggests that there could be a simple formula: for example $b_{10} = 2938057$, which is prime

We can show Result one by restriction to $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and impose $f(0) = 0$

Result 1. *$f_- = f_+$ are smooth functions with $f_- = f_+$ is an actual solution to the equation $f'(x) = e^{f^{-1}(x)}$, it is merly c^∞ but not analytic having divergent power series*

Proof. This idea has been explored above in **6**, where a formal power series expansion is obtained for f which does not seem to converge for any $x \neq 0$.

Taking another approach, we can use an iteration scheme starting from $f_1(x) = x$ and inductively solve the ODE $f'_{n+1} = e^{f_n^{-1}}$ with the initial condition $f_{n+1}(0) = 0$ to obtain f_{n+1}, much in the spirit of Picard iteration. Explicitly, for example, we have

$f'_2 = e^x$ and $f_2 = e^x - 1$;

$f'_3 = e^{\ln(x+1)} = 1 + x$ and $f_3 = x + x^2/2$;

$f'_4 = e^{\sqrt{1+2x}-1}$ and $f_4 = e^{\sqrt{1+2x}-1}(\sqrt{1+2x} - 1)$

and the next iteration produces non-elementary functions. It is clear that the sequence $(f_{2k-1})_{k\geq 1}$ is increasing, $(f_{2k})_{k\geq 1}$ is decreasing, and $f_{2k-1} < f_{2k}$, so there are respective limits $f_- = \lim_{k\to\infty} f_{2k-1}$ and $f_+ = \lim_{k\to\infty} f_{2k+1}$, with $f_- \leq f_+$. It is also clear that from $n \geq 2$ on the function $f'_n = e^{f_{n-1}^{-1}}$ is positive and increasing, so f_n is increasing and convex, which can be passed to the limit to show that both f_- and f_+ are also increasing and convex. As such they are continuous, and by **Dini's theorem , [(see the bellow footnote)]** f_{2k-1} converges to f_- locally uniformly and similarly for f_+ ,This is one of the few situations in mathematics where pointwise convergence implies uniform convergence; the key is the greater control implied by the monotonicity. Note also that the limit function must be continuous, since a uniform limit of continuous functions is necessarily continuous. Furthermore, the inequality $|x - y| \leq |f_n(x) - f_n(y)|$ (as $f'_n = e^{f_{n-1}^{-1}} \geq 1$) can also be passed to the limit. Then the following chain of inequalities: [3] $|f_-^{-1}(x) - f_{2k-1}^{-1}(x)| \leq |x - f_-(f_{2k-1}^{-1}(x))| = |f_{2k-1}(f_{2k-1}^{-1}(x)) - f_-(f_{2k-1}^{-1}(x))|$ shows that f_{2k-1}^{-1} converges locally uniformly to f_-^{-1}, which then implies f'_{2k} converges

[3] In the mathematical field of analysis, Dini's theorem says that if a monotone sequence of continuous functions converges pointwise on a compact space and if the limit function is also continuous, then the convergence is uniform ,The standard theorem is the following: Let $f_k\colon [a,b] \to \mathbb{R}$ be a sequence of functions, such that f_k is non-increasing (resp. non-decreasing) for every $k \in \mathbb{N}$. If (f_k) converges pointwise to a **continuous** function $f\colon [a,b] \to \mathbb{R}$, then f is non-increasing (resp. non-decreasing) and the convergence is uniform.

locally uniformly to $e^{f_-^{-1}}$. Hence $f_+' = e^{f_-^{-1}}$, and similarly $f_-' = e^{f_-^{-1}}$. From this it can be shown that f_{2k-1} converges to f_- locally in C^∞, so both f_- and f_+ are smooth functions, and they form an orbit of order at most 2 of the above iteration scheme. Moreover it can be shown that the first n terms of the Taylor expansion of f_n agrees with what have been calculated formally in 6 (preliminary analysis), so both f_- and f_+ have the same Taylor expansion as calculated using formal power series expansion.then we are done

Now if we attempt to solve the equation $f'(x) = e^{f^{-1}(x)}$ with f map $\mathbb{R} \to \mathbb{R}$ we w'd say:

Lemma 5. *There is no such function satisfies $f'(x) = e^{f^{-1}(x)}$ Since f would have to map $\mathbb{R} \to \mathbb{R}$.*

Proof. There is no such function. Since f would have to map $\mathbb{R} \to \mathbb{R}$ for the equation to make sense at all $x \in \mathbb{R}$, it follows that $f^{-1}(x) \to -\infty$ also as $x \to -\infty$, so $f' \to 0$. Thus $f(x) \geq x$, say, for all small enough x, hence $f^{-1}(x) \leq x$ eventually, but then the equation shows that $f' \leq e^x$, which is integrable on $(-\infty, 0)$, so f would approach a limit as $x \to -\infty$ and not be surjective after all.

Remark. :Now might one explore the idea and ask about analyticity of the equation with the domain restriction of f to be defined as $f : \mathbb{R}_+ \to \mathbb{R}_+$ and impose $f(0) = 0$, here the convergence is not hard for demonstrate as shown below , but the question about the analyticity at 0 seems to be less obvious.

Result 2. *The functional equation $f'(x) = e^{f^{-1}(x)}$ converges in L^1 and therefore in c^∞ for $x \geq 0$*

Proof. Assume f, h are two increasing functions with $f(0) = h(0) = 0$ and $f(x), h(x) \geq x$ and F, H are their image under the picard map ,then for every $T > 0$ the functional: $\Phi(f, h, T) = \int_0^T |f(t) - h(t)|\, dt$ satisfies $\Phi(F, H, T) \leq \int_0^T e^t \Phi(f, h, t)\, dt$ and it follows that on every finite interval $[0, T]$ we have convergence in L^1 and c^∞ ☐

Before going to present a general proof for partial results which include both :result (03) and (04) and also result (01) adding some detail for it ,we must show that the titled diff-functional has a unic solution in a formal power series which it is divergent ,The following lemma is a formel version of the standard proof of **Picard-Lindelf** .

Lemma 6. *Let k be some field. There is a formal differentiation in the ring of formal power series $k[[x]]$. Let $F(x, y) \in k[[x, y]]$ be a formal series which is algebraic over x and y. Consider a differential equation:*

$$y' = F(x, y)$$

where y belongs to the maximal ideal of $k[[x]]$, so $F(x, y)$ is well-defined

Proof. we Rewrite the desired condition as:

$$y = \int_0^x F(x, y)\, dx = \sum_{n,m \geq 0} f_{n,m} \int_0^x x^n y^m\, dx = L(x, y).$$

We compute that: $L(x, y_0) - L(x, y_1) = \sum_{n,m \geq 0} f_{n,m} \int_0^x x^n (y_0^n - y_1^m)\, dx$ hence that if $x^k | y_0 - y_1$ then $x^{k+1} | L(x, y_0) - L(x, y_1)$. It follows that the operation $y \mapsto L(x, y)$ on $xk[[x]]$ is Lipschitz with respect to the x-adic metric with Lipschitz constant less than 1 (the exact constant depends on how you're defining the x-adic metric), hence has a unique fixed point by the Banach fixed point theorem. (This is a formal version of the standard proof of Picard-Lindelf.) Moreover, this fixed point has coefficients in the field generated by $f_{n,m}$. By the way this is a very simple fact which is verified by hands. we just plug a formal power series for y, and see that all coefficients can be uniquely determined. (Condition that y belongs to the maxial ideal is just a fancy way to state that the constant term of y is zero, that is "$y(0) = 0$"). It is included in many old books on analytic functions and differential equations.For example H. Cartan [14]

[4]

[4]This solution is way too complicated. It's much easier: the way we teach students to find a power series solution. Just plug in and equate similar terms to get a "chain-like" linear system in the coefficients of y: each coefficient is uniquely found in terms of the previous ones. (And we assume $y_0 = 0$ Or, if we prefer, keep differentiating the equation and substituting Or, $x = 0$ we w'll get $y^{(n)}$ in terms of the previous derivatives. Since the series are formal, it's even easier as there's no convergence issue

Theorem 1. (Lagrange Inversion theorem) *If $y = f(x)$ with $f(a) = b$ and $f'(a) \neq 0$, then*

$$x(y) = a + \sum_{n=1}^{\infty} \left(\lim_{x \to a} \frac{d^{n-1}}{dx^{n-1}} \left(\frac{x-a}{f(x)-b} \right)^n \frac{(y-b)^n}{n!} \right).$$

Might someone ask for the derivation of LIF for complex ,just to check this paper[13] which it montioned the theorem with proof for reals and complex

8. Solution of $f'(x) = e^{f^{-1}(x)}$ in a formel power series:

[5] [6]

Definition 1. *A formal power series, sometimes simply called a "formal series" (Wilf 1994), of a field* $\boldsymbol{F}$ $[a_0, a_1, a_2, \cdots]$ *over* $\boldsymbol{F}$ *is an infinite sequence Equivalently, it is a function from the set of nonnegative integers to* $\boldsymbol{F}$ $[0, 1, 2, \cdots] \to \boldsymbol{F}$ *.A formal power series is often written :* $a_0 + a_1 x + a_2 x + \cdots + a_n x^n + \cdots$ *but with the understanding that no value is assigned to the symbol* x

Lemma 7. *The functional-equation* $f'(x) = e^{f^{-1}(x)}$ *has a unic divergent solution in formel power series*

Proof. A formal Taylor series (e.g.f.) solution about the origin can be obtained a few ways. Let $f^{(-1)}(x) = e^{b.x}$ with $(b.)^n = b_n$ and $b_0 = 0$, Then [07] (Bell polynomials) gives the e.g.f.

$$e^{f^{(-1)}(x)} = e^{e^{b.x}} = 1 + b_1 x + (b_2 + b_1^2)\frac{x^2}{2!} + (b_3 + 3b_1 b_2 + b_1^3)\frac{x^3}{3!} + \cdots ,$$

[5] We denote in the proof of **lemma 7** by e.g.f or E.G.F : The exponontial generating function and by o.g.f or O.G.F :The ordinary generation function and by LIF by The inversion lagrange formula [12] Which is presented below for reals.

[6] An exponential generating function (E.G.F) for the integer sequence $a_0, a_1 \cdots$ is a function $E(x)$ such that $E(x) = \sum_{k=0}^{+\infty} a_k \frac{x^k}{k!} = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + \cdots$, The ordinary generating function is a function associated with a sequence :$a_0, a_1 \cdots$ is a function whose value at x is $\sum_{i=0}^{+\infty} a_i x^i$

and the Lagrange inversion / series reversion formula (LIF) [08] gives

$$f'(x) = \frac{1}{b_1} + \frac{1}{b_1^3}(-b_2)x + \frac{1}{b_1^5}(3b_2^2 - b_1 b_3)\frac{x^2}{2!} + \cdots \ .$$

,Equating the two series and solving recursively gives:

$$b_n \to (0, 1, -1, 3, -16, 126, -1333, ...)$$

which is signed [09]. This follows from the application of the inverse function theorem (essentially the LIF again)

$$f'(z) = 1/f^{(-1)\prime}(\omega) \ ,$$

when $(z, \omega) = (f^{(-1)}(\omega), f(z))$, leading to

$$f^{(-1)\prime}(x) = \exp[-f^{(-1)}(f^{(-1)}(x))],$$

The differential equation defining signed [09], Applying the LIF to the sequence for b_n gives the e.g.f. $f(x) = e^{a.x}$ equivalent of F.C.'s o.g.f.

$$a_n \to (0, 1, 1, 0, 1, -6, 52, ...).$$

As another consistency check, we apply the formalism of [10] for finding the multiplicative inverse of an e.g.f. to find the e.g.f. for $\exp[-A(-x)] = \exp[f^{(-1)}(x)]$ from that for

$$\exp[A(-x)] = 1 - x + 2\frac{x^2}{2!} - 7\frac{x^3}{3!} + \cdots \ ,$$

which is signed [11], as noted in [09]. This gives $f'(x) = a.\ e^{a.x}$.

[7]

[7] The inverse function theorem here might be more aptly called the inverse formal series theorem. As we can see, the differential equations and inverses here in analytic guise are concise statements of relations among the coefficients of formal series (e.g.f.s or o.g.f.s), Really we are talked about the Lagrange inversion theorem such that we mean the plain derivation of coefficients of the series for $f^{(-1)}$ and also the residue formula $[x^n]f^{(-1)} = \frac{1}{n}\mathrm{Res}(f^{-n})$, for more informations we can check The July 2015 formula in [**A133437**], and we pay tribute to Lagrange by calling pretty much any formula an LIF, Think Lagrange inversion = compostitional inversion via series whether o.g.f.s, e.g.f.s, or other series reps. For non-series inversion, we might use directly $\boldsymbol{g(g^{-1}(x)) = x}$ or a Laplace-like transform with a change of variables ,Any way to skin the cat analytically but we don't have a well-defined analytic function, forward or inverse, to begin with here though, so bootstrap methods only come to mind.

Proof. :**General proof:** There is no analytic local solution at 0 to $f' = e^{f^{-1}}$, $f(0) = 0$, that is, the formal power series solution is diverging. , this means $f_- = f_+$ is an actual solution to the equation $f'(x) = e^{f^{-1}(x)}$ [**result (01)**], we shall consider the equivalent equation

$$\begin{cases} h' = e^{h \circ h}, \\ h(0) = 0, \end{cases}$$

satisfied by $h(x) := -f^{-1}(-x)$ (Indeed, by the rule of the derivative of an inverse, $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = e^{-f^{-1}(f^{-1}(x))}$ so that $h'(x) = e^{h(h(x))}$; see also the proof of **lemma 07**) Indeed, assume by contradiction the formal power series solution $x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{2}{3}x^4 + \&c.$ to the above equation has a positive radius of convergence. Then, it extends uniquely by analytic continuation to a maximally-defined analytic function, still denoted h (that is, defined on the largest positive interval $[0, a)$, for some $0 < a \leq +\infty$). Note that the Taylor series of h at 0 has non-negative coefficients. This follows immediately by induction, equating the coefficients of h' and $e^{h \circ h}$; incidentally, This series is the EGF of the positive integer sequence [09], As a consequence (check the details below), h is totally monotonic on $[0, a)$; in particular $h'(x) > h'(0) = 1$ and $h(x) > x$ for all $0 < x < a$, and h is invertible. [**result(03)**]
Then observe that $\log(h'(h^{-1}(x))$ is a well-defined analytic function on the interval $h[0, a)$, and coincides with h locally at 0. By the maximality of $[0, a)$ we have thus $h[0, a) \subset [0, a)$, but, due to the inequality $h(x) > x$ on $(0, a)$, this inclusion is only possible if $a = +\infty$, so that h is unbounded [**result(04)**]. On the other hand, since $e^{-h(h(t))}h'(t) = 1$ and $h(t) \geq t$, we have for any $x \geq 0$

$$x = \int_0^x e^{-h(h(t))}h'(t)dt = \int_0^{h(x)} e^{-h(s)}ds \leq \int_0^{+\infty} e^{-s}ds = 1,$$

a contradiction.

- **Note 01:** To justify the total monotonicity of h [**result(03)**], Note that, as a general elementary fact, a real analytic function on an interval I, whose Taylor series at some point $x_0 \in I$ has non-negative coefficients, has Taylor series with non-negative coefficients ay any point $x \in I$, $x \geq x_0$. Indeed, this is clear for $x_1 \geq x_0$ within the radius of convergence of x_0, and since there is a uniform radius of convergence at any $y \in [x_0, x]$, one reaches x by finitely many steps $x_0 < x_1 < \cdots < x_n = x$.

- **Note 2:** . The same argument works for other differential-functional equations like e.g.

$$\begin{cases} h' = 1 + h \circ h, \\ h(0) = 0, \end{cases}$$

that generates the sequence [OEIS A001028]. As before, a maximally-defined analytic solution h, if any, must be totally monotonic and defined for all $x \geq 0$, for otherwise $h' \circ h^{-1} - 1$ would be a proper extension of it. Then we reach a contradiction as before, with one more step needed: since we have $\frac{h'(t)}{1+h(h(t))} = 1$ and $h(t) \geq t$ for any $t \geq 0$, we also have, for any $x \geq 0$

$$x = \int_0^x \frac{h'(t)dt}{1 + h(h(t))} = \int_0^{h(x)} \frac{dt}{1 + h(t)} \leq \int_0^{h(x)} \frac{dt}{1 + t} = \log(1 + h(x)),$$

whence $e^x \leq 1 + h(x)$; if we plug this into the latter inequalities again, we get

$$x = \int_0^{h(x)} \frac{dt}{1 + h(t)} \leq \int_0^{h(x)} e^{-t} dt \leq 1,$$

as before. By comparison, the same conclusion also holds for $h' = F(h \circ h)$ with any F analytic and totally monotonic on $(-\epsilon, +\infty)$, and with $F(0) = 1$.

Remark 1. We deduced result **06** from calculation of the few terms of coefficients as shown above in **6** using **mathematica** , For result**05** really we can't able to check weither this function is Borel -summable f , see [15] (you can see bellow definition of Borel summation in footenote) , we see that f is smooth but not analytic and we don't know if this is an asymptotic series to which Borel summation could be applied, The derivatives of the n-folds iterate f^n of f have a curious formula such that : for any $n \in \mathbb{N}$: $(f^n)' = \exp(f^{-1} + f^0 + f^1 + \cdots + f^{n-2})$,$(f^{-n})' = \exp(-f^{-2} - f^{-3} - \cdots - f^{-n+1})$. for instance we try to check if the function we have asymptotics or no according to the following definitions .

[8] [9]

Definition 2. *A series :*$a_0 + a_1x + a_2x^2 + \cdots$ *is said to be an asymptotic for* $f(x)$ *, near* $x = 0$ *if : :*$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \mathcal{O}\left(x^{n+1}\right)$ *for each* n *and small* x

Definition 3. *The definition of asymptotics series is interesting only when the series is divergent , if* $f(x)$ *is regular at the origin [see footnote page 14] then it's Taylor series* $\sum_{n=0}^{\infty} a_nx^n$ *is convergent for small* x *and satisfy* ***definition 2*** *[see, 16,p28]*

The solution of the titled functional equation which is presented as a formal power series as shown above in general proof and **proof of lemma 7** and as noted in [09] which has the following form :$f(x) = x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{2}{3}x^4 +$ &c. is the exponential generating function converges only at $x = 0$ and has the positive radius of convergence also centred at the origin then it is **regular** hence the function we have satisfies both **definition 2** and **definition 3** then it is asymptotics but this is not enough to judge that is a Borel summable because it's not a well defined analytic function .The next definition will show to us that the function we have does not have analytic continuation to (a neighborhood of) $x = r$ such that r is the radius of convergence .

Definition 4. *. A series* $\sum_{n=0}^{+\infty} \frac{c_n}{z^{n+1}}$ *is Borel summable [17, def.3,page 17] for* $z > 0$ *if the series* $f(x) = \sum_{n=0}^{+\infty} b_n \frac{x^n}{n!}$ *has a radius of convergence* $R > 0$ *and if the function:*

$$f(x) = \sum_{n=0}^{+\infty} b_n \frac{x^n}{n!} \tag{1}$$

[8]In mathematics, Borel summation is a summation method for divergent series, introduced by mile Borel (1899). It is particularly useful for summing divergent asymptotic series, and in some sense gives the best possible sum for such series. There are several variations of this method that are also called Borel summation, and a generalization of it called Mittag-Leffler summation.

[9]The meaning of the word regular is not precisely defined. Sometimes they say regular enough which means (for instance) that a function is differentiable, or twice continuously differentiable and so on. Usually saying this they want the function to fulfill all the needed assumptions ,If the power series consists of powers of x, Then it means that the series has a positive radius of convergence. If the series is not centered at the origin(not powers of x but of $x - a$ for some $a \neq 0$) then it means that there is an analytic continuation to the origin that is regular at the origin

has an analytic continuation along $\mathbb{R}+$

with $\int_0^{+\infty} e^{-xz} g(x)dx$ *converges for* $z > 0$ *Then we define :*

$$\sum_{n=0}^{Borel} \frac{c_n}{z^{n+1}} = \int_0^{+\infty} e^{-xz} f(x)dx \qquad (2)$$

We show here that the radius of convergence of the function f defined in (1) must be positive for applying Borel- summation. Really the problem we are challenged in **Definition .4** is the convergence of the integral in the **R.H.S** of the equation (2), For the formel solution $f(x) = x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{2}{3}x^4 + \&c.$ the term b_n is positive for $n > 1$ **(see.result. (06))** and it is increasing sequence and satisfy :$b_n > n!$ for $n > 4$ as shown in the bellow table which signed the sequence A214645 in [09]: [10]

[10]Watson's theorem gives conditions for a function to be the Borel sum of its asymptotic series,and says that in this region f is given by the Borel sum of its asymptotic series. More precisely, the series for the Borel transform converges in a neighborhood of the origin, and can be analytically continued to the positive real axis, and the integral defining the Borel sum converges to $f(z)$ for z in this region $|z| < R$

n =n	b_n
$n = 1$	1
$n = 2$	1
$n = 3$	3
$n = 4$	16
$n = 5$	126
$n = 6$	1333
$n = 7$	17895
$n = 8$	293461
$n = 9$	5721390
$n = 10$	129948787
$n = 11$	384796695
$n = 12$	99848190706
$n = 13$	3301868304168
$n = 14$	121369298328835
$n = 15$	4923587573624940
$n = 16$	219090125559917698
$n = 17$	10637377855875861600
$n = 18$	560928617456424367993
$n = 19$	31993928581562975604588
$n = 20$	1966682218962058310721178

Remark 2. As $b_n > n!$ for $n > 4$, the series diverges at $x = 1$, hence its radius of convergence r lies in $[0, 1]$ precisely $r = 0$. Then, by [Pringheim's theorem] see [18] , $g(x)$ does not have analytic continuation to a neighborhood of $x = r$ Hence the integral can't converge for any x over $(0, +\infty)$.One of the few cases we can get the convergence of integral defined in **R.H.S** of (2) with $b_n > n!$ is the extension to complex plane , We take this as example :

$$b_n = (-1)^n n! \binom{1/2}{n}. \tag{3}$$

which gives :

$$\sum \frac{a_n x^n}{n!} = \sqrt{1 - x} \tag{4}$$

near zero, and does have an analytic continuation to $(1, \infty)$ in fact two of them through the complex plane. And the integral converges.

Recall that the defined formel solution of $f' = e^{f^{-1}}$ is :

$$f(x) = x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{2}{3}x^4 + \&c. \tag{5}$$

Now the **definition 4** show to us that the formel series solution defined in **3** can't be Borel- summable since $r = 0$ and must be positive wherease **definition 2** and **definition 3** showed that function could be Borel summable missing the bounded of the error term which is necessary condition to apply Watson's theorem [19 , (see also .footnote,page 15)], just we used the regularity of the formel series at the origin.In any way we arn't able to know if the function defined in (6) is Borel-summable in the context of all previews definitions. The special feature in the problem discussed here is the possibility to find such a function explicitly and to use it to find a formula for b_n, rather than that we don't have an explicit formula or closed form for b_n or b_n ([09]) hasn't any known recursive formula for example :$b_{n+1}+b_{n-1} = (n\alpha+\beta)b_n, n \geq 1.(\alpha, \beta)$ real or complex [11] Recurrence relations of this type appear in several contexts see[20] and [21] and also sequences :A053983, A053984, A058797, A058798, A058799. see[20, sloane 2008].The determination of the explicit formula for b_n by any linear reccurence is very important to get analytic solution satisfies the asymptotics expansion which is defined as :

$$\sum b_n x^n + \mathcal{O}\left(x^{n+1}\right) \tag{6}$$

for more explanation see the following remark

Remark 3. By a theorem of Borel (1895) [22], see also Carleman (1926, Ch. V) [23], given any sequence b_n there exists a C^∞ function on $\mathbb{R}$ with these numbers as Taylor coefficients at 0, and thus the asymptotic $\sum b_n z^n$ there; moreover, we may choose the function so that it is analytic in, e.g., a given sector expansion in D in the complex plane, with the given asymptotic expansion as $z \to 0$in D Hence, the existence of a function (and, indeed, infinitely many functions) representing a given sequence by an asymptotic expansion is well-known.

Now, since we are unable to define an explicit formula of the sequence defined in [09] for predicting the Borel sum of (6), we shall use the bellow theorem (**theorem.2**) which uses Borel's method .

[11]let $f(z) = \sum_{n\geq 0} a_n z^n$ be regular at the point O and let be the set of all its singular points. Draw the segment OP and the straight line L_p normal to OP through any point $P \in C$ The set of points on the same side with O for each straight line L_p is denoted by $\prod$ the boundary Γ of the domain $\prod$is then called the Borel polygon of the function $f(z)$.

9. Borel's Methods

if:

$$e^{-x}\sum_{A_n}\frac{x^n}{n!}\to A \tag{7}$$

we say that :$A_n \to A(B)$ and if :

$$\int_0^{+\infty} e^{-x}\sum_{a_n}\frac{x^n}{n!}dx = \lim_{X\to+\infty}\int_0^{X} e^{-x}\sum_{a_n}\frac{x^n}{n!}dx = A \tag{8}$$

we say that $A_n \to A(B')$.The methods are of quite different types , The first(7)being 'integral function' definition in the sense of **4.12** ,see ([16, page 79]) with $J(x) = e^x$ and the second (8) a "moment method in the sense of . **4.13**,see ([16,page 81]) with $\mu_n = n!$, $X(x) = 1 - e^{-x}$, but the special properties of the exponential function make them all but equivalent.for a short proof see [16, page 79] under " Method B and B' are regular . ,The following Theorem w'd be in the context of the cited Borel's method.

Theorem 2. *The power series representing a function regular at the origin is summable* (B') *inside the Borel polygon (see.the above .footnote) of the function , regulary and uniformaly throughout any closed region interior to the polygon ; and is not summable at any point outside the polygon.*

The present theorem which uses Borel's method is available to be active in complex plane then by extension from $\mathbb{R}+$ to $\mathbb{C}$ we have a well defined analytic function $f(z)$ since it is Holomorphic (smooth in $\mathbb{R}+$) in some region D in $\mathbb{C}$ more than that $f(z)$ is convergent only for $z = 0$ which means it has a positive radius of convergence and it is centred at the origin hence we have got a regular complex valued function at the origin which satisfies the above theorem , And do not forgot since it is regular at the origin then it has asymptotic series convergent for small z , Now from the given conditions for Borel- summable to be applied in the conext of the above theorem We are ready to present the following conjecture which include both real and complex plane .

conjecture 1. *The solution of the differential-functional* $f' = e^{f^{-1}}$ *which is presented in a formel power series as noted in [09] is Borel-summable* (B') *inside the Borel polygon of the function regulary and uniformaly throughout any closed region interior to the polygon ; and is not summable at any point outside the polygon in the complex plane and could be applied over* $\mathbb{R}+$

10. Conclusion:

Formal power series with radius of convergence 0 often arise in counting labeled graphs. For example, the exponential generating function for labeled connected graphs is $\log G(x)$, where

$$G(x) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!},$$

which has radius of convergence 0.

Series like $\sum_{n=0}^{\infty} n! x^n$ arise in the theory of continued fractions; this series has the continued fraction expansions

$$\cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{2x}{1 - \cfrac{2x}{1 - \cfrac{3x}{1 - \cfrac{3x}{1 - \cdots}}}}}}}$$

and

$$\cfrac{1}{1 - x - \cfrac{x^2}{1 - 3x - \cfrac{2^2 x^2}{1 - 5x - \cfrac{3^2 x^2}{1 - 7x - \cdots}}}}$$

Similar continued fractions exist for ordinary generating functions (with radius of convergence 0) for Bell numbers, Eulerian polynomials, matchings, and more generally, moments of orthogonal polynomials. A very nice combinatorial approach to these continued fractions has been given by Philippe Flajolet, Combinatorial aspects of continued fractions[24]. It is true that most, if not all, of these examples of nonconverging power series can be refined to power series in more than one variable that do converge for some values of the parameters. For example, the exponential generating function for labeled connected graphs by edges is $\log G(x,t)$, where

$$G(x,t) = \sum_{n=0}^{\infty} (1+t)^{\binom{n}{2}} \frac{x^n}{n!};$$

this converges for $|1+t|<1$. On the other hand, the exponential generating function for strongly connected tournaments is $1-1/G(x)$, and this doesn't seem to generalize since $1-1/G(x,t)$ has some negative coefficients.Particulary the solution of the titled functional is smooth function but not analytic in $\mathbb{R}+$ then the existence of this kind of functions represents one of the main differences between differential geometry and analytic geometry. In terms of sheaf theory, this difference can be stated as follows: the sheaf of differentiable functions on a differentiable manifold is fine, in contrast with the analytic case.probably there is some one find any rigorous application of this function in sheaf theory .

Aknowledgements: I would like to thank Pietro Majer and Tom copland for some contributions to this paper, And also zeraoulia Elhadj for helping us to improve the quality of this paper .and i'm willing for any comments.

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