

Question 430 : A Definite Integral

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abstract

This note presents a definite integral for pi

1. INTRODUCTION. The constant pi is defined by

$$\pi = 4 \int_0^1 \frac{1}{1+x^2} dx = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.14159265... \quad (1)$$

2. A DEFINITE INTEGRAL FOR PI.

$$\begin{aligned} \frac{\pi}{4} + 1 = & \int_0^{4/27} \cosh^{-1} \left(\frac{2}{\sqrt{3x}} \cos \left(\frac{1}{3} \cos^{-1} \frac{3\sqrt{3x}}{2} \right) \right) dx + \\ & + \int_{4/27}^2 \cosh^{-1} \left(\sqrt[3]{\frac{1}{2x} + \frac{1}{x} \sqrt{\frac{1}{4} - \frac{1}{27x}}} + \sqrt[3]{\frac{1}{2x} - \frac{1}{x} \sqrt{\frac{1}{4} - \frac{1}{27x}}} \right) dx \end{aligned} \quad (2)$$

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