Lemniscate Constants

(*Second Version*)

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abstract

This note presents some integrals for lemniscate constants.

Keywords: Lemniscate constants,integrals,gamma function,number Pi.

Introduction

1. The first Lemniscate constant is given by (Ref.6,p. 502,Eq. 19.20.2)

$$L_1=\frac{\left(\Gamma\left(1/4\right)\right)^2}{4\sqrt{2\pi}}=\int_0^1\frac{1}{\sqrt{1-x^4}}dx \tag{1}$$

2. The second Lemniscate constant is given by (Ref.6,p. 503,Eq. 19.20.22)

$$L_2=\frac{\left(\Gamma\left(3/4\right)\right)^2}{\sqrt{2\pi}}=\int_0^1\frac{x^2}{\sqrt{1-x^4}}dx \tag{2}$$

3. Lemniscatic Identity (Euler ~1781)

$$\int_0^1\frac{1}{\sqrt{1-x^4}}dx\times\int_0^1\frac{x^2}{\sqrt{1-x^4}}dx=\frac{\pi}{4} \tag{3}$$

Integrals for L_1

$$L_1=1+\int_1^\infty\left(1-\sqrt[4]{1-x^{-2}}\right)dx \tag{4}$$

$$L_1=\int_1^\infty\frac{1}{\sqrt{x^4-1}}dx \tag{5}$$

$$L_1 = \int_0^\infty \left(\sqrt[4]{1+x^{-2}} - 1 \right) dx \tag{6}$$

$$L_1 = \frac{1}{2} \int_0^1 \frac{1}{\sqrt{x\left(1-x^2\right)}} dx \tag{7}$$

$$L_1 = \frac{1}{2} \int_1^\infty \frac{1}{\sqrt{x\left(x^2-1\right)}} dx \tag{8}$$

$$L_1 = \frac{3}{4} \int_0^1 \frac{1}{\sqrt[4]{x}\sqrt{1-x^3}} dx \tag{9}$$

$$L_1 = \int_0^{\ln\left(1+\sqrt{2}\right)} \frac{1}{\sqrt{1-\left(\sinh x\right)^2}} dx \tag{10}$$

$$L_1 = \int_0^\infty \frac{1}{\sqrt{1+\left(\cosh x\right)^2}} dx \tag{11}$$

$$L_1 = \int_0^\infty \sqrt{\frac{1-\left(\tanh x\right)^2}{1+\left(\tanh x\right)^2}} dx \tag{12}$$

$$L_1 = \int_0^{1/\sqrt{2}} \cosh^{-1}\left(\sqrt{x^{-2}-1} \right) dx \tag{13}$$

$$L_1 = \frac{1}{2} \int_0^1 \cosh^{-1}\left(\frac{1}{x^2} \right) dx \tag{14}$$

$$L_1 = \int_0^1 \tanh^{-1}\left(\sqrt{\frac{1-x^2}{1+x^2}} \right) dx \tag{15}$$

$$L_1 = \frac{1}{2\sqrt{2}} \int_0^\infty \sinh^{-1}\left(\frac{1}{x^2} \right) dx \tag{16}$$

$$L_1 = \int_0^{1/\sqrt{2}} \ln\left(\sqrt{x^{-2}-1}+\sqrt{x^{-2}-2}\right)dx \tag{17}$$

$$L_1 = \frac{1}{\sqrt{2}}\left(1+\frac{\ln 2}{2}\right)+\int_0^{1/\sqrt{2}} \ln\left(\sqrt{1-x^2}+\sqrt{1-2x^2}\right)dx \tag{18}$$

$$L_1 = \frac{1}{2\sqrt{2}}\int_0^{\infty} \frac{\sinh^{-1} x^2}{x^2}\,dx \tag{19}$$

$$L_1 = \frac{1}{2}\int_1^{\infty} \frac{\cosh^{-1} x^2}{x^2}\,dx \tag{20}$$

$$L_1 = \int_0^{\infty} \frac{\sqrt[4]{1+x^2}-1}{x^2}\,dx \tag{21}$$

$$L_1 = \int_0^{\infty} \frac{1}{\left(x+\sqrt[4]{x^4+x^2}\right)\left(x+\sqrt{x^2+1}\right)}\,dx \tag{22}$$

$$L_1 = 1+\int_0^{1} \frac{1-\sqrt[4]{1-x^2}}{x^2}\,dx \tag{23}$$

$$L_1 = 1+\int_1^{\infty} \frac{1}{\left(x+\sqrt[4]{x^4-x^2}\right)\left(x+\sqrt{x^2-1}\right)}\,dx \tag{24}$$

$$L_1 = \int_0^{\pi/2} \frac{1}{\sqrt{1+(\sin x)^2}}\,dx \tag{25}$$

$$L_1 = \frac{\pi}{2\sqrt{2}}+\int_{1/\sqrt{2}}^{1} \sin^{-1}\left(\sqrt{x^{-2}-1}\right)dx \tag{26}$$

$$L_1 = \frac{\pi}{2\sqrt{2}}+\int_1^{\sqrt{2}} \frac{1}{x^2}\sin^{-1}\left(\sqrt{x^2-1}\right)dx \tag{27}$$

$$L_1 = \frac{1}{2}\int_0^1 \left(\sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-x}{1+x}} \right) \frac{dx}{\sqrt{1+x^2}} \tag{28}$$

$$L_1 = \frac{\pi}{4} + \frac{1}{2}\int_1^\infty \left(\frac{\pi}{2} - \cos^{-1}\left(\frac{1}{x^2} \right) \right) dx \tag{29}$$

$$L_1 = \frac{\pi}{4} + \frac{1}{2}\int_1^\infty \sin^{-1}\left(\frac{1}{x^2} \right) dx \tag{30}$$

$$L_1 = \frac{\pi}{4} + \frac{1}{2}\int_0^1 \frac{\sin^{-1} x^2}{x^2}\, dx \tag{31}$$

$$L_1 = \frac{\pi}{2\sqrt{2}} + \int_0^1 \frac{x \sin^{-1} x}{\left(1+x^2\right)^{3/2}}\, dx \tag{32}$$

$$L_1 = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{x(1-x)(2-x)}}\, dx \tag{33}$$

$$L_1 = \int_0^\infty \frac{1}{\sqrt{e^{2x} - e^{-2x}}}\, dx = \frac{1}{\sqrt{2}}\int_0^\infty \frac{1}{\sqrt{\sinh(2x)}}\, dx \tag{34}$$

$$L_1 = \frac{3}{4}\int_0^1 \frac{1}{\sqrt{x\left(3-3x+x^2\right)}\,\sqrt[4]{1-x}}\, dx \tag{35}$$

$$L_1 = \int_0^1 \frac{1}{\sqrt{x(2-x)\left(2-2x+x^2\right)}}\, dx \tag{36}$$

$$L_1 = \int_0^\infty \frac{1}{\sqrt{x(2+x)\left(2+2x+x^2\right)}}\, dx \tag{37}$$

$$L_1 = \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{x}\left(1+x^2\right)^{3/4}}\, dx \tag{38}$$

$$L_1 = \frac{1}{2}\int_1^\infty \frac{1}{\sqrt{x}\left(x^2-1\right)^{3/4}}\, dx \tag{39}$$

$$L_1 = \int_1^{\sqrt{2}} \frac{1}{\sqrt{\left(2-x^2\right)\left(x^2-1\right)}}\,dx \tag{40}$$

$$L_1 = \int_{1/\sqrt{2}}^{1} \frac{1}{\sqrt{\left(2x^2-1\right)\left(1-x^2\right)}}\,dx \tag{41}$$

$$L_1 = \frac{1}{2}\int_0^1 \left(1-x^2\right)^{-3/4} dx \tag{42}$$

$$L_1 = \frac{1}{2} + \frac{1}{2}\int_1^{\infty} \left(1-\sqrt{1-x^{-4/3}}\right) dx \tag{43}$$

$$L_1 = \int_0^1 \frac{1}{\sqrt{\left(1-x^2\right)\left(2-x^2\right)}}\,dx \tag{44}$$

$$L_1 = \int_0^{\infty} \frac{1}{\sqrt{\cosh^2 x + \sinh^2 x}}\,dx = \int_0^{\infty} \frac{1}{\sqrt{\cosh\left(2x\right)}}\,dx \tag{45}$$

$$L_1 = \int_0^{\pi/4} \frac{1}{\sqrt{\cos^2 x - \sin^2 x}}\,dx = \int_0^{\pi/4} \frac{1}{\sqrt{\cos\left(2x\right)}}\,dx \tag{46}$$

$$L_1 = \int_0^{\pi/2} \left(\cos^2 x + 2\sin^2 x\right)^{-1/2} dx \tag{47}$$

$$L_1 = \frac{1}{\sqrt{2}} + \int_{1/\sqrt{2}}^{\infty} \left(1-\sqrt{\frac{3-\sqrt{1+4x^{-2}}}{2}}\right) dx \tag{48}$$

$$L_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}} \frac{\sqrt{2}-\sqrt{3-\sqrt{1+4x^2}}}{x^2}\,dx \tag{49}$$

$$L_1 = \frac{1}{2}\int_0^1 \left(\sqrt{\frac{1+x^2}{1-x^2}} + \sqrt{\frac{1-x^2}{1+x^2}}\right) dx \tag{50}$$

$$L_1 = \frac{1}{4}\int_0^1 \left(\sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-x}{1+x}} \right) \frac{dx}{\sqrt{x}} \tag{51}$$

$$L_1 = \frac{1}{4}\int_0^{\pi/2} \left(\tan\frac{x}{2} + \cot\frac{x}{2} \right) \frac{\sin x}{\sqrt{\cos x}}\, dx \tag{52}$$

$$L_1 = \frac{1}{4}\int_0^{\pi/2} \left(\tan\frac{x}{2} + \cot\frac{x}{2} \right) \sqrt{\sec x - \cos x}\, dx \tag{53}$$

$$L_1 = \frac{1}{2}\int_1^{\infty} \frac{x^{1/2} + x^{-1/2}}{(1+x)\sqrt{x^2-1}}\, dx \tag{54}$$

$$L_1 = 2 - \frac{3}{2}\int_0^1 \sqrt[4]{\frac{1-x}{(1+x)^7}}\, dx \tag{55}$$

$$L_1 = 2 - \frac{3}{2}\int_0^1 \sqrt[4]{\frac{x}{(2-x)^7}}\, dx \tag{56}$$

$$L_1 = \frac{1}{2}\int_0^1 \left(x(2-x) \right)^{-3/4} dx \tag{57}$$

$$L_1 = \frac{3}{2}\int_0^1 \sqrt{\frac{x}{1-x^6}}\, dx \tag{58}$$

$$L_1 = 1 + \int_0^1 \frac{x\sqrt{x}}{1-x^2+\sqrt{1-x^2}}\, dx \tag{59}$$

$$L_1 = 1 + \int_0^1 \frac{x^4}{1-x^4+\sqrt{1-x^4}}\, dx \tag{60}$$

$$L_1 = \frac{1}{4}\int_1^{\infty} \left(x^2 - x \right)^{-3/4} dx \tag{61}$$

$$L_1 = \int_0^{\infty} \left(2 - \sqrt{2}\sqrt{\sqrt{2\sqrt{4+\left(f(x)\right)^2} - f(x)} - \sqrt{f(x)}} \right) dx \tag{62}$$

$$f(x)=\sqrt[3]{\sqrt{\frac{64}{27}+\frac{1}{4x^8}}+\frac{1}{2x^4}}-\sqrt[3]{\sqrt{\frac{64}{27}+\frac{1}{4x^8}}-\frac{1}{2x^4}} \tag{63}$$

$$\begin{aligned} L_1 &= \frac{1}{2}\int_0^u \left(-\frac{1}{2}+\sqrt[3]{\frac{1}{4x^2}+\sqrt{\frac{1}{16x^4}-\frac{1}{12^3}}}+\sqrt[3]{\frac{1}{4x^2}-\sqrt{\frac{1}{16x^4}-\frac{1}{12^3}}}\right)dx+ \\ &\quad +\frac{1}{2}\int_u^\infty \left(-\frac{1}{2}+\frac{1}{\sqrt{3}}\cos\left(\frac{1}{3}\cos^{-1}\frac{6\sqrt{3}}{x^2}\right)\right)dx \end{aligned} \tag{64}$$

$$u=\sqrt[4]{108} \tag{65}$$

Integrals for L_2

$$L_2=\int_0^\infty\left(1-\sqrt[4]{\frac{x^2}{1+x^2}}\right)dx \tag{66}$$

$$L_2=\int_1^\infty \frac{1}{x^2\sqrt{x^4-1}}\,dx \tag{67}$$

$$L_2=\frac{1}{2}\int_0^1\sqrt{\frac{x}{1-x^2}}\,dx \tag{68}$$

$$L_2=\frac{1}{2}\int_0^1\sqrt{\frac{1-x}{x(2-x)}}\,dx \tag{69}$$

$$L_2=\frac{1}{2}\int_1^\infty \frac{1}{x\sqrt{x\left(x^2-1\right)}}\,dx \tag{70}$$

$$L_2=\int_0^{\ln\left(1+\sqrt{2}\right)} \frac{\left(\sinh x\right)^2}{\sqrt{1-\left(\sinh x\right)^2}}\,dx \tag{71}$$

$$L_2 = \int_{\ln(1+\sqrt{2})}^{\infty} \frac{1}{\sinh^2 x\sqrt{\sinh^2 x - 1}}\,dx \tag{72}$$

$$L_2 = \int_0^{\infty} \frac{1}{(\cosh x)^2\sqrt{1+(\cosh x)^2}}\,dx \tag{73}$$

$$L_2 = \int_0^{\infty} (\tanh x)^2\sqrt{\frac{1-(\tanh x)^2}{1+(\tanh x)^2}}\,dx \tag{74}$$

$$L_2 = \frac{1}{2}\int_0^{\infty} \sqrt{\tanh x}\sqrt{1-\tanh^2 x}\,dx \tag{75}$$

$$L_2 = \frac{1}{2}\int_0^{\infty} \sqrt{\tanh x}\,\operatorname{sech} x\,dx \tag{76}$$

$$L_2 = \frac{1}{2}\int_0^{\infty} (\operatorname{sech} x)^{3/2}\,dx \tag{77}$$

$$L_2 = \frac{1}{2}\int_0^{\pi/2} \sqrt{\cos x}\,dx \tag{78}$$

$$L_2 = \frac{1}{2}\int_0^{\pi/2} \sqrt{\sin x}\,dx \tag{79}$$

$$L_2 = \int_0^{\pi/4} \frac{\tan^2 x}{\sqrt{\cos(2x)}}\,dx \tag{80}$$

$$L_2 = \int_0^{\pi/2} \frac{(\sin x)^2}{\sqrt{1+(\sin x)^2}}\,dx \tag{81}$$

$$L_2 = \int_0^{\pi/2} \frac{\cos^2 x}{\sqrt{1+\cos^2 x}}\,dx \tag{82}$$

$$L_2 = \int_0^{1/\sqrt{2}} \cos^{-1}\left(\sqrt{\frac{x^2 + x\sqrt{4+x^2}}{2}}\right) dx \tag{83}$$

$$L_2 = \frac{\pi}{2\sqrt{2}} - \int_0^{1/\sqrt{2}} \sin^{-1}\left(\sqrt{\frac{x^2 + x\sqrt{4+x^2}}{2}}\right) dx \tag{84}$$

$$L_2 = \int_0^{\pi/2} \left(2\cos^2 x + \sin^2 x\right)^{-3/2} \sin^2 x\, dx \tag{85}$$

$$L_2 = \frac{1}{2}\int_0^1 \left(\sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-x}{1+x}}\right) \frac{x^2}{\sqrt{1+x^2}}\, dx \tag{86}$$

$$L_2 = \frac{\sqrt[4]{27}}{4\sqrt{2}} + \frac{1}{4}\int_u^\infty \left(1 - \cos\left(\frac{1}{3}\cos^{-1}\left(1 - \frac{27}{2x^4}\right)\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{1}{3}\cos^{-1}\left(1 - \frac{27}{2x^4}\right)\right)\right) dx \tag{87}$$

$$u = \sqrt[4]{27} / \sqrt{2}$$

$$L_2 = \int_0^{1/\sqrt{2}} \cosh^{-1}\left(\sqrt{-\frac{1}{3} + \sqrt[3]{\frac{1}{2x^2} - \frac{1}{27} + \frac{1}{x}\sqrt{\frac{1}{4x^2} - \frac{1}{27}}} + \sqrt[3]{\frac{1}{2x^2} - \frac{1}{27} - \frac{1}{x}\sqrt{\frac{1}{4x^2} - \frac{1}{27}}}}\right) dx \tag{88}$$

Final Integrals

$$L_1 + L_2 = \int_0^1 \sqrt{\frac{1+x^2}{1-x^2}}\, dx \tag{89}$$

$$L_1 + L_2 = \frac{1}{2}\int_0^1 \sqrt{\frac{1+x}{x(1-x)}}\, dx \tag{90}$$

$$L_1 - L_2 = \int_0^1 \sqrt{\frac{1-x^2}{1+x^2}}\, dx \tag{91}$$

$$L_1 - L_2 = \frac{1}{2}\int_0^1 \sqrt{\frac{1-x}{x(1+x)}}\, dx \tag{92}$$

$$L_1 - L_2 = \int_0^\infty \frac{1}{1+x^2+\sqrt{1+6x^2+x^4}}\, dx \tag{93}$$

$$L_1 + L_2 = 1 + \int_1^\infty \left(1 - \sqrt{\frac{x^2-1}{x^2+1}}\right) dx \tag{94}$$

$$L_1 + L_2 = \frac{1+\sqrt{2}}{2} + \frac{1}{2}\int_{1+\sqrt{2}}^\infty \left(1 - \frac{\sqrt{1-6x^2+x^4}}{x^2}\right) dx \tag{95}$$

$$L_1 + L_2 = \int_0^{\pi/2} \sqrt{1+\sin^2 x}\, dx \tag{96}$$

$$L_1 - L_2 = \int_0^{\pi/2} \frac{\cos^2 x}{\sqrt{1+\sin^2 x}}\, dx \tag{97}$$

$$L_1 + L_2 = \frac{\pi}{\sqrt{2}} - \int_1^{\sqrt{2}} \sin^{-1}\sqrt{x^2-1}\, dx \tag{98}$$

$$L_1 - L_2 = \int_0^1 \sin^{-1}\left(\sqrt{\frac{2+x^2-x\sqrt{8+x^2}}{2}}\right) dx \tag{99}$$

$$L_1 + L_2 = \int_0^{\pi/2} \sqrt{2-\cos^2 x}\, dx \tag{100}$$

$$L_1 + L_2 = \frac{\pi}{\sqrt{2}} - \int_1^{\sqrt{2}} \cos^{-1}\sqrt{2-x^2}\, dx \tag{101}$$

$$L_1 - L_2 = \frac{1}{2}\int_0^{\pi/2} \frac{\sin x}{\sqrt{\cos x}} \tan\left(\frac{x}{2}\right) dx \tag{102}$$

$$L_1 - L_2 = \frac{1}{2}\int_0^{\pi/2} \sqrt{\sec x - \cos x}\ \tan\left(\frac{x}{2}\right) dx \tag{103}$$

$$L_1 + L_2 = \frac{1}{2}\int_0^{\pi/2} \frac{\sin x}{\sqrt{\cos x}} \cot\left(\frac{x}{2}\right) dx \tag{104}$$

$$L_1 + L_2 = \frac{1}{2}\int_0^{\pi/2} \sqrt{\sec x - \cos x}\ \cot\left(\frac{x}{2}\right) dx \tag{105}$$

$$L_1 + L_2 = \int_1^{\infty} \frac{1}{x^2}\sqrt{\frac{x^2+1}{x^2-1}}\, dx \tag{106}$$

$$L_1 - L_2 = \int_1^{\infty} \frac{1}{x^2}\sqrt{\frac{x^2-1}{x^2+1}}\, dx \tag{107}$$

$$L_1 + L_2 = \int_0^{\infty} \frac{\sqrt{\cosh^2 x + 1}}{\cosh^2 x}\, dx \tag{108}$$

$$L_1 - L_2 = \int_0^{\infty} \frac{\tanh^2 x}{\sqrt{\cosh^2 x + 1}}\, dx \tag{109}$$

$$L_1 + L_2 = \int_0^{\ln\left(1+\sqrt{2}\right)} \frac{\cosh^2 x}{\sqrt{1 - \sinh^2 x}}\, dx \tag{110}$$

$$L_1 - L_2 = \int_0^{\ln\left(1+\sqrt{2}\right)} \sqrt{1 - \sinh^2 x}\, dx \tag{111}$$

$$L_1 + L_2 = \int_0^{\infty} \sqrt{1 + \tanh^2 x}\, \operatorname{sech} x\, dx \tag{112}$$

$$L_1 + L_2 = \int_0^{\infty} \sqrt{1 - \tanh^4 x}\, dx \tag{113}$$

$$L_1 + L_2 = \int_0^{1} \tanh^{-1} \sqrt[4]{1 - x^2}\, dx \tag{114}$$

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