Lemniscate Constants

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abstract
This note presents some integrals for lemniscate constants.

Introduction

1. The first Lemniscate constant is given by

$$L_1 = \frac{\left(\Gamma(1/4)\right)^2}{4\sqrt{2\pi}} = \int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx \tag{1}$$

2. The second Lemniscate constant is given by

$$L_2 = \frac{\left(\Gamma(3/4)\right)^2}{\sqrt{2\pi}} = \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx \tag{2}$$

3. Lemniscatic Identity (Euler ~1781)

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx \times \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx = \frac{\pi}{4} \tag{3}$$

Integrals for L_1

$$L_1 = \int_1^\infty \left(1 - \sqrt[4]{1-x^{-2}}\right)dx \tag{4}$$

$$L_1 = \int_1^\infty \frac{1}{\sqrt{x^4-1}}\,dx \tag{5}$$

$$L_1 = \int_0^\infty \left(\sqrt[4]{1+x^{-2}} - 1\right)dx \tag{6}$$

$$L_1 = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{x\left(1-x^2\right)}}\,dx \tag{7}$$

$$L_1 = \frac{1}{2}\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx \tag{8}$$

$$L_1 = \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{1+\sqrt{1+4x^{-2}}}{2}}-1\right)dx \tag{9}$$

$$L_1 = \int_0^{\ln\left(1+\sqrt{2}\right)} \frac{1}{\sqrt{1-\left(\sinh x\right)^2}}\,dx \tag{10}$$

$$L_1 = \int_0^\infty \frac{1}{\sqrt{1+\left(\cosh x\right)^2}}\,dx \tag{11}$$

$$L_1 = \int_0^\infty \sqrt{\frac{1-\left(\tanh x\right)^2}{1+\left(\tanh x\right)^2}}\,dx \tag{12}$$

$$L_1 = \int_0^{1/\sqrt{2}} \cosh^{-1}\left(\sqrt{x^{-2}-1}\right)dx \tag{13}$$

$$L_1 = \int_1^\infty \left(\ln\left(1+\sqrt{2}\right)-\sinh^{-1}\left(\sqrt{1-x^{-2}}\right)\right)dx \tag{14}$$

$$L_1 = \int_0^1 \tanh^{-1}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)dx \tag{15}$$

$$L_1 = \int_1^\infty \ln\left(\frac{1+\sqrt{2}}{\sqrt{1-x^{-2}}+\sqrt{2-x^{-2}}}\right)dx \tag{16}$$

$$L_1 = \int_0^{1/\sqrt{2}} \ln\left(\sqrt{x^{-2}-1}+\sqrt{x^{-2}-2}\right)dx \tag{17}$$

$$L_1 = \frac{1}{\sqrt{2}}\left(1+\frac{\ln 2}{2}\right)+\int_0^{1/\sqrt{2}} \ln\left(\sqrt{1-x^2}+\sqrt{1-2x^2}\right)dx \tag{18}$$

$$L_1 = \int_1^{\infty} \ln\left(\frac{\left(1+\sqrt{2}\right)x}{\sqrt{x^2-1}+\sqrt{2x^2-1}}\right)dx \tag{19}$$

$$L_1 = \int_0^1 \frac{1}{x^2}\ln\left(\frac{1+\sqrt{2}}{\sqrt{1-x^2}+\sqrt{2-x^2}}\right)dx \tag{20}$$

$$L_1 = \int_0^{\infty} \frac{\sqrt[4]{1+x^2}-1}{x^2}dx \tag{21}$$

$$L_1 = \int_0^{\infty} \frac{1}{\left(x+\sqrt[4]{x^4+x^2}\right)\left(x+\sqrt{x^2+1}\right)}dx \tag{22}$$

$$L_1 = \int_0^1 \frac{1-\sqrt[4]{1-x^2}}{x^2}dx \tag{23}$$

$$L_1 = \int_1^{\infty} \frac{1}{\left(x+\sqrt[4]{x^4-x^2}\right)\left(x+\sqrt{x^2-1}\right)}dx \tag{24}$$

$$L_1 = \int_0^{\pi/2} \frac{1}{\sqrt{1+\left(\sin x\right)^2}}dx \tag{25}$$

$$L_1 = \frac{\pi}{2\sqrt{2}}+\int_{1/\sqrt{2}}^{1} \sin^{-1}\left(\sqrt{x^{-2}-1}\right)dx \tag{26}$$

$$L_1 = \frac{\pi}{2\sqrt{2}}+\int_1^{\sqrt{2}} \frac{1}{x^2}\sin^{-1}\left(\sqrt{x^2-1}\right)dx \tag{27}$$

$$L_1 = \frac{1}{2}\int_0^1 \left(\sqrt{\frac{1+x}{1-x}}+\sqrt{\frac{1-x}{1+x}}\right)\frac{dx}{\sqrt{1+x^2}} \tag{28}$$

$$L_1=\int_0^{\infty}\left(2-\sqrt{2}\sqrt{\sqrt{2\sqrt{4+\left(f\left(x\right)\right)^2}-f\left(x\right)}-\sqrt{f\left(x\right)}}\right)dx \tag{29}$$

$$f\left(x\right)=\sqrt[3]{\sqrt{\frac{64}{27}+\frac{1}{4x^8}}+\frac{1}{2x^4}}-\sqrt[3]{\sqrt{\frac{64}{27}+\frac{1}{4x^8}}-\frac{1}{2x^4}} \tag{30}$$

$$\begin{aligned}L_1=\frac{1}{2}\int_0^{u}\left(-\frac{1}{2}+\sqrt[3]{\frac{1}{4x^2}+\sqrt{\frac{1}{16x^4}-\frac{1}{12^3}}}+\sqrt[3]{\frac{1}{4x^2}-\sqrt{\frac{1}{16x^4}-\frac{1}{12^3}}}\right)dx+\\+\frac{1}{2}\int_u^{\infty}\left(-\frac{1}{2}+\frac{1}{\sqrt{3}}\cos\left(\frac{1}{3}\cos^{-1}\frac{6\sqrt{3}}{x^2}\right)\right)dx\end{aligned} \tag{31}$$

$$u=\sqrt[4]{108} \tag{32}$$

Integrals for L_2

$$L_2=\int_0^{\infty}\left(1-\sqrt[4]{\frac{x^2}{1+x^2}}\right)dx \tag{33}$$

$$L_2=\int_1^{\infty}\frac{1}{x^2\sqrt{x^4-1}}dx \tag{34}$$

$$L_2=\frac{1}{2}\int_0^{1}\sqrt{\frac{x}{1-x^2}}\,dx \tag{35}$$

$$L_2=\int_0^{\ln\left(1+\sqrt{2}\right)}\frac{\left(\sinh x\right)^2}{\sqrt{1-\left(\sinh x\right)^2}}dx \tag{36}$$

$$L_2=\int_0^{\infty}\frac{1}{\left(\cosh x\right)^2\sqrt{1+\left(\cosh x\right)^2}}dx \tag{37}$$

$$L_2 = \int_0^{\infty} \left(\tanh x\right)^2 \sqrt{\frac{1-\left(\tanh x\right)^2}{1+\left(\tanh x\right)^2}}\, dx \tag{38}$$

$$L_2 = \int_0^{\pi/2} \frac{\left(\sin x\right)^2}{\sqrt{1+\left(\sin x\right)^2}}\, dx \tag{39}$$

$$L_2 = \frac{\pi}{2\sqrt{2}} - \int_0^{1/\sqrt{2}} \sin^{-1}\left(\sqrt{\frac{x^2 + x\sqrt{4+x^2}}{2}}\right) dx \tag{40}$$

$$L_2 = \frac{1}{2}\int_0^1 \left(\sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-x}{1+x}}\right) \frac{x^2}{\sqrt{1+x^2}}\, dx \tag{41}$$

$$L_2 = \frac{\sqrt[4]{27}}{4\sqrt{2}} + \frac{1}{4}\int_u^{\infty} \left(1-\cos\left(\frac{1}{3}\cos^{-1}\left(1-\frac{27}{2x^4}\right)\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{1}{3}\cos^{-1}\left(1-\frac{27}{2x^4}\right)\right)\right) dx \tag{42}$$

$$u = \sqrt[4]{27} / \sqrt{2}$$

$$L_2 = \int_0^{1/\sqrt{2}} \cosh^{-1}\left(\sqrt{-\frac{1}{3} + \sqrt[3]{\frac{1}{2x^2} - \frac{1}{27} + \frac{1}{x}\sqrt{\frac{1}{4x^2} - \frac{1}{27}}} + \sqrt[3]{\frac{1}{2x^2} - \frac{1}{27} - \frac{1}{x}\sqrt{\frac{1}{4x^2} - \frac{1}{27}}}}\right) dx \tag{43}$$

Final Integrals

$$L_1 + L_2 = \int_0^1 \sqrt{\frac{1+x^2}{1-x^2}}\, dx \tag{44}$$

$$L_1 + L_2 = \frac{\sqrt{2}+1}{2} + \frac{1}{2}\int_{\sqrt{2}+1}^{\infty} \frac{6x^2 - 1}{x^2\left(x^2 + \sqrt{1-6x^2+x^4}\right)}\, dx \tag{45}$$

$$L_1 - L_2 = \int_0^1 \sqrt{\frac{1-x^2}{1+x^2}}\, dx \tag{46}$$

$$L_1 - L_2 = \int_0^{\infty} \frac{1}{1+x^2+\sqrt{1+6x^2+x^4}}\,dx \tag{47}$$

References

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