A Geometric-Probabilistic problem about the lengths of the segments intersected in straights that randomly cut a triangle

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A Geometric-Probabilistic problem about the lengths of the segments intersected in straights that randomly cut a triangle.

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Abstract. If a line cuts randomly two sides of a triangle, the length of the segment determined by the points of intersection is also random. The object of this study, applied to a particular case, is to calculate the probability that the length of such segment is greater than a certain value.

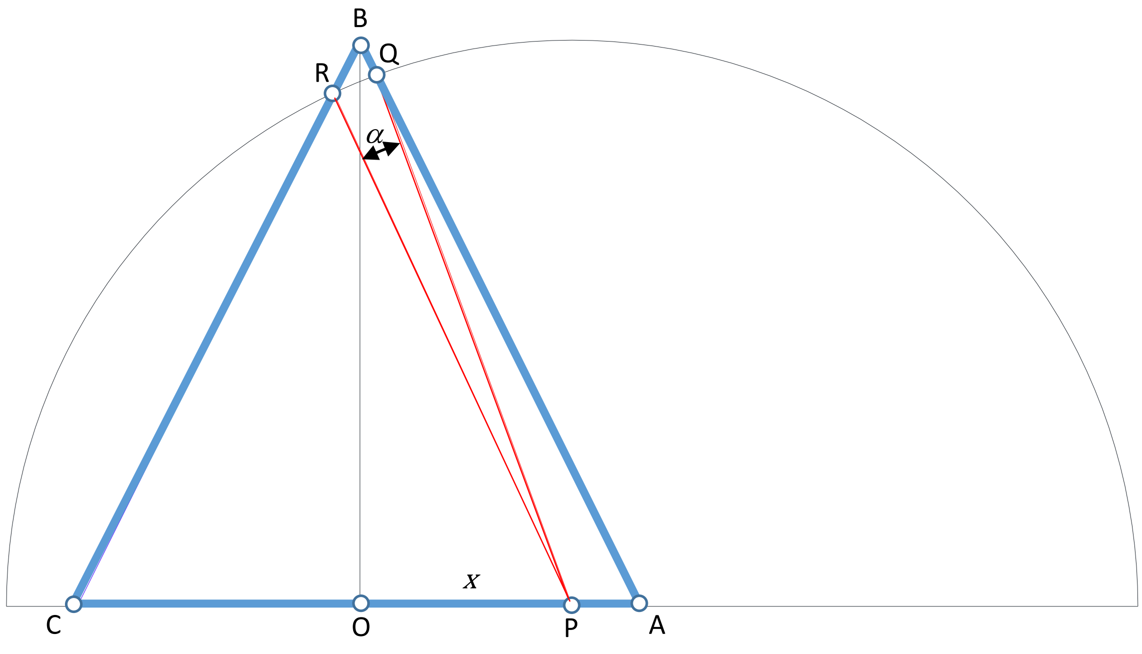
Let ABC be an isosceles triangle, with $\overline{AB} = \overline{CB}$ and $\overline{OB} = \overline{AC}$, being O the midpoint of $\overline{AC}$ (ie, $\overline{OB}$ is the height relative to the side $\overline{AC}$).

Through a randomly chosen point P on $\overline{AC}$ is drawn a straight r with also randomly chosen slope. Let Q and R be the points where r intersects $\overline{AB}$ and $\overline{CB}$, respectively.

Calculate the probability for the following inequalities:

$$\boxed{\overline{PQ} > \overline{AC} \ or \ \overline{PR} > \overline{AC}} \qquad (1)$$

Let us draw an arc of radius $\overline{AC}$ with center P. Let P and Q be the intersection points of this arc with the sides $\overline{AB}$ y $\overline{CB}$, respectively, as shown in the following picture, with the triangle represented in an orthonormal coordinate system, with origin at O, x-axis (abscissas) in the direction OA and y-axis (ordinates) in the direction OB.



Clearly, all the straight lines of the bundle with vertex P in $\overline{AC}$ intersect the sides $\overline{AB}$ or $\overline{CB}$, and all the lines of the sub-bundle inner to the angle $\alpha = \widehat{QPR}$, and only them, satisfies (1).

Since x and α are **continuous** random variables **uniformly distributed**, for a differential of length dx in $\overline{AC}$, the probability that the condition (1) is satisfied will be

$$\boxed{dp = \frac{\alpha}{\pi} dx} \qquad (2)$$

and therefore, the probability that the inequalities (1) are satisfied for a randomly chosen point in $\overline{AC}$ will be

$$\boxed{p = \frac{1}{\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \alpha(x)\, dx} \qquad (3)$$

where $\alpha(x)$ is the function relating the angle α with the abscissa x.

We have used the following facts:

- In (2):
  - In an infinitesimal length, dx, the limit angle α is **constant.**
  - The slope of the secant line is **independent** of the abscissa x.

- In (3):
  - The required probability p is obtained by Riemann integration of the **probability density function** $\alpha(x)$ in the symmetric interval $\overline{AC} = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

The limit angle α can be expressed in radians as:

$$\boxed{\alpha = \pi - \widehat{QPA} - \widehat{CPR}} \qquad (4)$$

But,

$$\overset{\triangle}{QPA}: \boxed{\widehat{QPA} = \pi - \hat{A} - \hat{Q}} \qquad (5)$$

$$\overset{\triangle}{CPR}: \boxed{\widehat{CPR} = \pi - \hat{C} - R} \qquad (6)$$

So,

$$\boxed{\alpha = \hat{Q} + \hat{R} + \hat{A} + \hat{C} - \pi} \qquad (7)$$

Perhaps the easiest way to define α as a function of x is trigonometrically:

$$\overset{\triangle}{QPA}: \boxed{\frac{sin\hat{A}}{\overline{PQ}} = \frac{sin\hat{Q}}{\overline{AP}} \Rightarrow sin\hat{Q} = \frac{\overline{AP}}{\overline{PQ}} sin\hat{A}} \qquad (8)$$

$$\overset{\triangle}{CPR}: \boxed{\frac{sin\hat{C}}{\overline{PR}} = \frac{sin\hat{R}}{\overline{CP}} \Rightarrow sin\hat{R} = \frac{\overline{CP}}{\overline{PR}} sin\hat{C}} \qquad (9)$$

But $\hat{C} = \hat{A}$ and $tan\hat{A} = 2$. Moreover, without loss of generality, we can assume that

$$\boxed{\overline{AC} = \overline{OB} = 1} \tag{10}$$

Hence,

$$\boxed{sin\hat{A} = sin\hat{C} = \frac{2}{\sqrt{5}}} \tag{11}$$

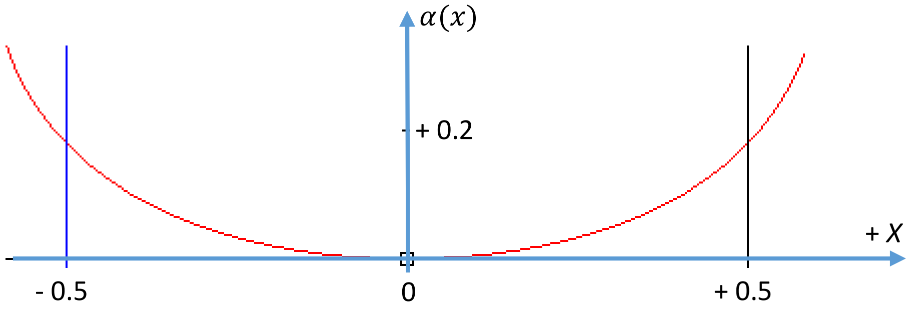
Applying (10) and (11) to (8) and (9), these reduce to

$$\boxed{sin\hat{Q} = \frac{1-2x}{\sqrt{5}}} \tag{8'}$$

$$\boxed{sin\hat{R} = \frac{1+2x}{\sqrt{5}}} \tag{9'}$$

Substituting in (7) this results we get the **probability density function** for the random variable α:

$$\boxed{\alpha(x) = arcsin\left(\frac{1-2x}{\sqrt{5}}\right) + arcsin\left(\frac{1+2x}{\sqrt{5}}\right) + 2arctan(2) - \pi} \tag{12}$$



Now, substituting in (3) the result given in (12), we obtain,

$$\boxed{p = \frac{1}{\pi}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} arcsin\left(\frac{1-2x}{\sqrt{5}}\right)dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} arcsin\left(\frac{1+2x}{\sqrt{5}}\right)dx\right] + \frac{2}{\pi}arctan(2) - 1} \tag{13}$$

These integrals (in indefinite form) can be solved by *integration by parts*. Let

$$\boxed{I_1 = \int arcsin\left(\frac{1-2x}{\sqrt{5}}\right)dx} \tag{14}$$

$$\boxed{I_2 = \int arcsin\left(\frac{1+2x}{\sqrt{5}}\right)dx} \tag{15}$$

$$I_1 = udv\left\{\begin{array}{l} u = arcsin\left(\frac{1-2x}{\sqrt{5}}\right) \\ dv = dx \end{array}\right\} \Rightarrow \left\{\begin{array}{l} du = \frac{-\frac{2}{\sqrt{5}}}{\sqrt{1-\left(\frac{1-2x}{\sqrt{5}}\right)^2}}dx \\ v = x \end{array}\right\}$$

And applying the **formula of integration by parts**,

$$I_1 = uv - \int vdu = x\, arcsin\left(\tfrac{1-2x}{\sqrt{5}}\right) - \int \frac{-\frac{2}{\sqrt{5}}x}{\sqrt{1-\left(\frac{1-2x}{\sqrt{5}}\right)^2}}\,dx \qquad (16)$$

Let

$$I_3 = \int \frac{-\frac{2}{\sqrt{5}}x}{\sqrt{1-\left(\frac{1-2x}{\sqrt{5}}\right)^2}}\,dx \qquad (17)$$

After simplifying the sub-integral expression, through the elementary transformations shown below, I_3 is reduced to two *quasi-immediate integrals* (reducible to immediate integrals by simple adjustment of constants). Omitting integration constants, for simplicity:

$$I_3 = \int \frac{-xdx}{\sqrt{-x^2+x+1}} = \int \frac{-2x+1-1}{2\sqrt{-x^2+x+1}}dx = \int \frac{-2x+1}{2\sqrt{-x^2+x+1}}dx - \int \frac{dx}{2\sqrt{-x^2+x+1}},$$

$$I_3 = \sqrt{-x^2+x+1} - \int \frac{dx}{2\sqrt{-x^2+x+1}} \qquad (18)$$

Let

$$I_4 = \int \frac{dx}{2\sqrt{-x^2+x+1}} \qquad (19)$$

$$I_4 = \int \frac{dx}{\sqrt{-4x^2+4x+4}} = \int \frac{dx}{\sqrt{5-(1-2x)^2}} = \int \frac{\frac{1}{\sqrt{5}}dx}{\sqrt{1-\left(\frac{1-2x}{\sqrt{5}}\right)^2}} = -\frac{1}{2}\int \frac{\frac{-2}{\sqrt{5}}dx}{\sqrt{1-\left(\frac{1-2x}{\sqrt{5}}\right)^2}}$$

$$I_4 = -\frac{1}{2} arcsin\left(\tfrac{1-2x}{\sqrt{5}}\right) \qquad (20)$$

From (18) and (19), $I_3 = \sqrt{-x^2+x+1} - I_4$; substituting in this the result given by (20),

$$I_3 = \sqrt{-x^2+x+1} + \frac{1}{2} arcsin\left(\tfrac{1-2x}{\sqrt{5}}\right) \qquad (21)$$

From (16), $I_1 = x\, arcsin\left(\frac{1-2x}{\sqrt{5}}\right) - I_3$, and substituting therein the result given by (21),

$$I_1 = \left(x - \tfrac{1}{2}\right) arcsin\left(\tfrac{1-2x}{\sqrt{5}}\right) - \sqrt{-x^2+x+1} \qquad (22)$$

And by a procedure completely analogously, we obtain

$$I_2 = \left(x + \tfrac{1}{2}\right) arcsin\left(\tfrac{1+2x}{\sqrt{5}}\right) + \sqrt{-x^2-x+1} \qquad (23)$$

Substituting in (13) this results, we obtain the exact value of the requested probability:

$$p = \frac{1}{\pi}\Big[I_1 + I_2\Big]_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{2}{\pi}arctan(2) - 1 = \frac{1}{\pi}\left[2arctan\left(\frac{1}{3}\right) + \frac{\pi}{2} + 1 - \sqrt{5}\right] + \frac{2}{\pi}arctan(2) - 1$$

$$p = \frac{2}{\pi}\left[arctan\left(\frac{1}{3}\right) + arctan(2)\right] - \frac{\sqrt{5}-1}{\pi} - \frac{1}{2} \qquad (24)$$

The expression (24) can be simplified considering the definition of the *golden ratio* [1] and the following identity regarding tangent arcs (by the general shape established in [2] for the decomposition of pi / 4 in two arctan):

$$arctan(2) = arctan\left(\frac{1}{3}\right) + \frac{\pi}{4} \qquad (25)$$

This identity can be proven easily by the formula of the tangent of a sum or through algebra of complex numbers, expressing the product of two complex numbers (suitably chosen) in two representation forms, binary form and polar form, as shown below.

Product in **binary form** and its corresponding representation in **polar form**:

$$(3+i)(1+i) = (2+4i) \;\Leftrightarrow\; \sqrt{10}_{arctan\left(\frac{1}{3}\right)}\sqrt{2}_{\frac{\pi}{4}} = \sqrt{20}_{arctan(2)}$$

After performed the product in polar form, the identity (25) is derived by identifying the arguments on both sides of the last equality:

$$\sqrt{20}_{arctan\left(\frac{1}{3}\right)+\frac{\pi}{4}} = \sqrt{20}_{arctan(2)} \qquad (26)$$

Finally, the result (24) can be expressed in the following elegant form that involves two of the most remarkable numbers: the **number Pi** and the **Golden Ratio** Φ,

$$\Phi = \frac{1+\sqrt{5}}{2} \qquad (27)$$

As the number π, it is surprising the ubiquity of this number, that emerge in the most diverse sceneries [1].

$$p = \frac{2}{\pi}\left(2\arctan\frac{1}{3} - \frac{1}{\Phi}\right) \qquad (28)$$

The approximate value of p in ten thousandths is, $p \approx 0.0162$.

References.

[1] Livio, Mario (2002). *The Golden Ratio: The Story of Phi, The World's Most Astonishing Number.* New York: Broadway Books. ISBN 0-7679-0815-5.

[2] Tanton, James (2012). *Mathematics Galore!. The First Five Years of the St. Mark's Institute of Mathematics.* The Mathematical Association of America. Washington. ISBN 978-0-88385-776-2.