Is Dark Matter and Black-Hole Cosmology an Effect of Born's Reciprocal Relativity Theory ?

Carlos Castro
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, Georgia. 30314, perelmanc@hotmail.com

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Abstract

Born's Reciprocal Relativity Theory (BRRT) based on a maximal proper-force, maximal speed of light velocity, inertial and non-inertial observers is re-examined in full detail. Relativity of locality and chronology are natural consequences of this theory, even in flat phase space. The advantage of BRRT is that Lorentz invariance is preserved and there is no need to introduce Hopf algebraic deformations of the Poincare algebra, de Sitter algebra, nor noncommutative spacetimes. After a detailed study of the notion of *generalized* force, momentum and mass in phase space, we explain that what one may interpret as "dark matter" surrounding galaxies, for example, is just an effect of observing ordinary galactic matter in *different accelerating* frames of reference than ours. Explicit calculations are provided that explain these novel relativistic effects due to the *accelerated* expansion of the Universe, and which generate the present-day density parameter value $\Omega_{DM} \sim 0.25$ of dark matter. The physical origins behind the numerical coincidences in Black-Hole Cosmology are also explored. We finalize with a rigorous study of the curved geometry of (co) tangent bundles (phase space) within the formalism of Finsler geometry, and provide a short discussion on Hamilton spaces.

Keywords : Born Reciprocity; Phase Spaces; Maximal Acceleration; Gravity; Dark Matter; Finsler Geometry.

1 Born's Reciprocal Relativity in Phase Space and Maximal Proper Force

Most of the work devoted to Quantum Gravity has been focused on the geometry of spacetime rather than phase space per se. The first indication that phase space should play a role in Quantum Gravity was raised by [1]. The principle of Born's reciprocal relativity [1] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality).

We explored in [4] some novel consequences of Born's reciprocal Relativity theory in flat phase-space and generalized the theory to the curved spacetime scenario. We provided, in particular, *six* specific results resulting from Born's reciprocal Relativity and which are *not* present in Special Relativity. These are : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations.

The generalized velocity and acceleration boosts (rotations) transformations of the $flat$ $8D$ Phase space, where $X^i, T, E, P^i; i = 1,2,3$ are *all* boosted (rotated) into each-other, were given by [2] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$. The $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\ dt \wedge dp_0 + \delta_{ij} dx^i \wedge dp^j; i,j = 1,2,3$ and also the following Born-Green line interval in the $flat$ $8D$ phase-space (in natural units $\hbar = c = 1$)

$$(d\omega)^2 \ = \ (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 + \frac{1}{b^2} \left((dE)^2 \ - \ (dp_x)^2 \ - \ (dp_y)^2 \ - \ (dp_z)^2 \right) \tag{1.1}$$

the rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate, see [2] for details. Born's reciprocity within the context of the conformal group $SU(2,2) \subset U(2,2)$ in $4D$ was explored by [3].

These transformations can be simplified drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions y, z, p_y, p_z intact. There is now a subgroup $U(1,1) = SU(1,1) \times U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 \ = \ (dT)^2 \ - \ (dX)^2 \ + \ \frac{(dE)^2 \ - \ (dP)^2}{b^2} =$$

$$(d\tau)^2 \ \left(\ 1 \ + \ \frac{(dE/d\tau)^2 \ - \ (dP/d\tau)^2}{b^2} \ \right) = (d\tau)^2 \ \left(\ 1 \ - \ \frac{F^2}{F^2_{max}} \ \right) \tag{1.2}$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in (2.2). The proper force interval $(dE/d\tau)^2 - (dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $(dT/d\tau)^2 - (dX/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in eq-(1.2) involves the ratios of two proper *forces*.

If (in natural units $\hbar = c = 1$) one sets the maximal proper-force to be given by $b \equiv m_P A_{max}$, where $m_P = (1/L_P)$ is the Planck mass and $A_{max} = (1/L_p)$, then $b = (1/L_P)^2$ may also be interpreted as the maximal string tension. The units of b would be of $(mass)^2$. In the most general case there are four scales of time, energy, momentum and length that can be constructed from the three constants $b, c, \hbar$ as follows

$$\lambda_t \;=\; \sqrt{\frac{\hbar}{bc}};\;\; \lambda_l \;=\; \sqrt{\frac{\hbar\ c}{b}};\;\; \lambda_p \;=\; \sqrt{\frac{\hbar\ b}{c}};\;\; \lambda_e \;=\; \sqrt{\hbar\ b\ c} \qquad (1.3)$$

The gravitational constant can be written as $G = \alpha_G\ c^4/b$ where α_G is a dimensionless parameter to be determined experimentally. If $\alpha_G = 1$, then the four scales in eq-(1.3) coincide with the *Planck* time, length, momentum and energy, respectively.

The $U(1,1)$ group transformation laws of the phase-space coordinates X, T, P, E which leave the interval (1.2) invariant are [2]

$$T' \;=\; T\ cosh\xi \;+\; (\frac{\xi_v\ X}{c^2} \;+\; \frac{\xi_a\ P}{b^2})\ \frac{sinh\xi}{\xi} \qquad (1.4a)$$

$$E' \;=\; E\ cosh\xi \;+\; (-\xi_a\ X \;+\; \xi_v P)\ \frac{sinh\xi}{\xi} \qquad (1.4b)$$

$$X' \;=\; X\ cosh\xi \;+\; (\xi_v\ T \;-\; \frac{\xi_a\ E}{b^2})\ \frac{sinh\xi}{\xi} \qquad (1.4c)$$

$$P' \;=\; P\ cosh\xi \;+\; (\frac{\xi_v\ E}{c^2} \;+\; \xi_a\ T)\ \frac{sinh\xi}{\xi} \qquad (1.4d)$$

ξ_v is the velocity-boost rapidity parameter and the ξ_a is the force (acceleration) boost rapidity parameter of the primed-reference frame. These parameters ξ_a, ξ_v, ξ are defined respectively in terms of the velocity $v = dX/dT$ and force $f = dP/dT$ (related to acceleration) as

$$tanh(\frac{\xi_v}{c}) = \frac{v}{c};\quad tanh(\frac{\xi_a}{b}) \;=\; \frac{F}{F_{max}},\;\; \xi \;=\; \sqrt{(\frac{\xi_v}{c})^2 + (\frac{\xi_a}{b})^2} \qquad (1.5)$$

It is straightforwad to verify that the transformations (1.4) leave invariant the phase space interval $c^2(dT)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = dT^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - c^2(dP)^2]$. Only the *combination*

$$(d\omega)^2 \;=\; (d\tau)^2\ \left(\ 1 \;-\; \frac{F^2}{F^2_{max}}\ \right) \qquad (1.6)$$

is truly left invariant under force (acceleration) boosts (1.4). They also leave invariant the symplectic 2-form (phase space areas) $\Omega = -\ dT \wedge E + dX \wedge dP$.

One can verify that the transformations eqs-(1.4) are invariant under the discrete transformations

$$(T, X) \rightarrow (E, P); \quad (E, P) \rightarrow (-T, -X), \quad \mathbf{b} \rightarrow \frac{1}{\mathbf{b}} \tag{1.7}$$

we argued [18] that the latter transformation $\mathbf{b} \rightarrow \frac{1}{\mathbf{b}}$ is a manifestation of the large/small tension T-duality symmetry in string theory. In natural units of $\hbar = c = 1$, the maximal proper force $\mathbf{b}$ has the same dimensions as a string tension (energy per unit length) $(mass)^2$.

1.1 Relativity of Locality and Chronology in Flat Phase Space

An immediate consequence of the transformations (1.4) is the *relativity of locality*. For example, for pure force/acceleration boosts $\xi = \xi_a/b, \xi_v = 0$, eqs-(1.4) reduce to

$$T' = T\ cosh\xi\ +\ \frac{P}{b}\ sinh\xi \tag{1.8a}$$

$$E' = E\ cosh\xi\ -b\ X\ sinh\xi \tag{1.8b}$$

$$X' = X\ cosh\xi\ -\ \frac{E}{b}\ sinh\xi \tag{1.8c}$$

$$P' = P\ cosh\xi\ +b\ T\ sinh\xi \tag{1.8d}$$

Consequently, given a local event in a given reference frame represented by the intersection of two worldlines, associated with two particles of equal mass, but *different* energies and momenta at the the point M, such that

$$X_1 = X_2,\ T_1 = T_2, \quad \Rightarrow \Delta X = \Delta T = 0 \tag{1.9}$$

In the primed reference frame as seen by an accelerated observer, we will have

$$\Delta T' = T_1' - T_2' = \frac{(P_1 - P_2)}{b}\ sinh\xi \neq 0, \quad P_1 \neq P_1 \tag{1.10a}$$

$$\Delta X' = X_1' - X_2' = -\frac{(E_1 - E_2)}{b}\ sinh\xi \neq 0, \quad E_1 \neq E_2 \tag{1.10a}$$

Therefore, what was a local event in a given reference frame is *not* local in the accelerated frame of reference when the two particles have *different* energy and momenta. This is what is meant by *relativity of locality*. In order to have $\Delta T' = \Delta X' = 0$ it would require to have in addition the conditions $E_1 = E_2, P_1 = P_2$; i.e. intersection of the worldlines in *phase space*.

Another way of viewing this phenomenon is by showing explicitly that given two intersecting world lines in one frame one can find an accelerated frame where they do not intersect.

Take particle 1 located at $X_1 = 0$, and particle 2 at $X_2 = L > 0$ at time $T = 0$, with different masses, moving with constant but different (positive) speeds, and such that

$$v_1 > v_2 > 0, \quad m_1 < m_2, \quad E_1 < E_2, \quad P_1 < P_2 \tag{1.11}$$

Due to the fact that particle 1 (with lower mass) is faster than particle 2 it will catch up with particle 2 at the time $T_* = L/(v_1 - v_2)$, and at the location $X_* = v_1 L/(v_1 - v_2) > L$. Therefore, the two worldlines will *intersect* at (X_*, T_*). The intersection of worldlines occurs because the faster particle is located to the left of the slower particle.

Let us evaluate the velocities in the accelerated frame when $\xi > 0$. From eqs-(1.8) one learns that

$$dX' \ = \ dX \ cosh\xi \ - \ \frac{dE}{b} \ sinh\xi \tag{1.12a}$$

and

$$dT' \ = \ dT \ cosh\xi \ + \ \frac{dP}{b} \ sinh\xi \tag{1.12b}$$

when E, P are *constant*, eqs-(1.12) yield

$$dX' \ = \ dX \ cosh\xi, \quad dT' \ = \ dT \ cosh\xi \ \Rightarrow \ v' = \frac{dX'}{dT'} = \frac{dX}{dT} \ = \ v \tag{1.13}$$

thus $v'_1 = v_1 > v'_2 = v_2 > 0$, and particle 1 remains faster than particle 2 in the accelerated frame. A careful inspection reveals then that *no* intersection of the two worldlines occurs in the accelerated frame of reference when the acceleration boost parameter $\xi > 0$ is sufficiently large to obey the condition

$$X'_2 \ = \ L \ cosh\xi \ - \ \frac{E_2}{b} \ sinh\xi \ < \ X'_1 \ = \ - \ \frac{E_1}{b} \ sinh\xi \ \Rightarrow$$

$$tanh\xi \ > \ \frac{L \ b}{E_2 - E_1} \ > \ 0, \quad E_2 > E_1, \quad \xi > 0 \tag{1.14}$$

This latter condition just states that in the accelerated frame of reference, for times $T' \geq 0$, the slower moving particle 2 will always remain to the left of the faster moving particle 1, $X'_2 < X'_1$. Since both particles have positive velocities, particle 2 will never catch up with particle 1, and consequently, their worldlines will *never* intersect in the accelerated frame of reference for $T' > 0$.

The worldlines actually intersect in the *past* $T'_* < 0$, in the moment when particle 1 crosses over particle 2 at $X'_* < 0$. Consequently, besides relativity of *locality*, we have also relativity of *chronology*. One observer will describe as a physical event to be one defined by the intersection of two worldlines taken place in his (her) *future*, an accelerated observer will describe it as an intersection of two worldlines taken place in his (her) *past*.

1.2 Other Geometric Models of Relative Locality

We finalize this section by following very *closely* a concise discussion of the relative locality geometry found recently in the introduction by [6]. Relative locality [7] is a framework originated from some interpretational issues connected to the possibility that energy-momentum space be curved, arisen in several contexts, as for example doubly special relativity (DSR) [8] , some models of noncommutative geometry [9] and 3D quantum gravity [10].

It is better understood now that the Planck-scale modifications of the particle dispersion relations can be encoded in the nontrivial geometrical properties of momentum space [7]. When both spacetime curvature and Planck-scale deformations of momentum space are present, it is expected that the nontrivial geometry of momentum space and spacetime get intertwined. The interplay between spacetime curvature and non-trivial momentum space effects was essential in the notion of "relative locality" and in the deepening of the relativity principle [7].

The theory is based on the assumption that physics takes place in phase space and there is no invariant global projection that gives a description of physical processes in spacetime. Therefore, local observers can construct descriptions of particles interacting in spacetime, but different observers construct different spacetimes, which correspond to different foliations of phase space. So, the notion of locality becomes observer dependent, whence the name of the theory.

This formulation of relative locality is very $different$ than ours despite the fact that both rely on the geometry of phase-spaces. Our results above are based on the nontrivial transformation properties of the coordinates X, T, E, P of $flat$ phase-spaces under force/acceleration boost transformations which *mix* spacetime coordinates with energy-momentum coordinates. Whereas the formulations [7], [8], [9], [10] rely on the geometry of *curved* phase-spaces, and the use of Hopf algebras leading to a deformed Poincare algebra, modified dispersion relations, a coproduct of momenta, and a coproduct of Lorentz generators.

We recall that DSR introduces in special relativity a new fundamental scale with the dimension of mass (usually identified with the Planck mass) in addition to the speed of light. The new scale gives rise to $deformations$ of the action of the Lorentz group on phase space, and consequently of the dispersion law of particles, of the addition law of momenta, and so on. Although doubly special relativity is mainly concerned with energy-momentum space, it is often realized in terms of *noncommutative* geometries that postulate a noncommutative structure of spacetime with a fundamental length scale of the order of the Planck length, and are in some sense dual to the DSR approach.

The energy-momentum space geometry defined in [7] has been investigated in a specific instance in [14], where it has been applied to the case of the κ-Poincare model [9], one of the favorite realization of DSR. This is a model of noncommutative geometry displaying a $deformed$ action of the Lorentz group on spacetime, whose energy-momentum space can be identified with a *curved* hyperboloid embedded in a 5-dimensional flat space [11].

The theory of relative locality refines this picture, by introducing some additional structures in the geometry of energy-momentum space, related to the properties of the *deformed* addition law of momenta, due to the coproduct of momenta associated with the Hopf algebraic structure, as for example, its (lack of) commutativity or associativity. The authors [6] investigated a different example of noncommutative geometry, namely the Snyder model [12] and its generalizations [13]. The distinctive property of this class of models is the preservation of the *linear* action of the Lorentz algebra on spacetime. This implies that the leading-order corrections to the composition law of the momenta must be cubic in the momenta, rather than quadratic. Moreover, the composition law is not only *noncommutative* but also *nonassocative*.

A new proposal [15] for the notion of spacetime in a relativistic generalization of special relativity based on a modification of the composition law of momenta has recently been presented. Locality of interactions is the principle which defines the spacetime structure for a system of particles. The main result [15] has been to show that it is possible to define a noncommutative spacetime for particles participating in an interaction in such a way that the interaction is seen as local for every observer. There exists then a large freedom to introduce a noncommutative spacetime in a relativistic theory beyond Special Relativity (SR) in a way compatible with the locality of interactions. An interesting particular case is the one in which the new spacetime of the two-particle system is such that the coordinates of one of the particles depend only on its own momentum.

To sum up, in the proposal [15], a noncommutative spacetime emerges in fact from a locality condition in a classical model which generalizes SR instead of from the implementation of a possible minimal length in a quantum spacetime. One may note that a maximal proper force does not necessarily imply a minimum length. Given $F = mc^2/L = b$ maximal proper force, one could have the scenario where $m \to 0, L \to 0$ such that $(m/L)c^2 = b$, and consequently there is no minimal length but there is a maximal proper force.

2 Generalized Mass, Momentum, Force and Dark Matter/Energy

2.1 Casimirs of $U(3,1)$ and Modified Dispersion Relations

One can verify by inspection of eqs-(1.8) that $E^2 - P^2$ is *not* invariant under force/acceleration boost transformations. Therefore the dispersion relation $E^2 - P^2 = m^2 \neq (E')^2 - (P')^2 = (m')^2$, and consequently the mass parameters m, m', are no longer invariant. This is not surprising since the quadratic Casimir of the Poincare algebra $P_\mu P^\mu = m^2$ is not the same as the quadratic Casimir of the pseudo-unitary algebra $U(1,3)$ [2]. In the case of a four-dim phase space, one has the following $U(1,1)$ quadratic Casimir

$$\mathcal{C}_2 \;=\; (\frac{T}{\lambda_t})^2 \;-\; (\frac{X}{\lambda_l})^2 \;+\; (\frac{E}{\lambda_e})^2 \;-\; (\frac{P}{\lambda_p})^2 \tag{3.1}$$

where we explicitly re-inserted the four scales of time, energy, momentum and length of eq-(1.3) to make $\mathcal{C}_2$ dimensionless.

If the temporal and spatial displacements are represented by the energy and momentum operators $E \rightarrow \frac{\partial}{\partial T}, P \rightarrow \frac{\partial}{\partial X}$, (in units of $\hbar = c = 1$), the Born reciprocity principle dictates that the energy and momentum displacements should be represented by the time and position operators $T \rightarrow \frac{\partial}{\partial E}, X \rightarrow \frac{\partial}{\partial P}$. Therefore we shall choose to define our $U(1,1)$ quadratic Casimir to be the following

$$\mathcal{C}_2 \;=\; \mathbf{M}^2 \;=\; b^2\;(T^2 \;-\; X^2) \;+\; E^2 \;-\; P^2, \;\; \hbar = c = 1 \tag{3.2}$$

and expressed in terms of the quantity $\mathbf{M} \neq m$, which has the same physical units of mass.

Given, $d\omega = d\tau\;\sqrt{1 - \frac{F^2}{b^2}}$, and $\mathbf{M}$, the *generalized* momentum in flat phase space is defined as

$$\mathcal{P}^M \;\equiv\; \mathbf{M}\;\frac{dZ^M}{d\omega} \;\;=\; \mathbf{M}\;\left(\;\frac{dT}{d\omega},\;\frac{dX}{d\omega},\;\frac{dE}{d\omega},\;\frac{dP}{d\omega}\;\right) \tag{3.3}$$

note that we have not explicitly inserted b^{-1} factors into the definition of $\mathcal{P}^M$, thus not all quantities in $\mathcal{P}^M$ have the same units. We shall re-insert these factors when we evaluate the norm

$$\mathcal{P}_M\;\mathcal{P}^M \;=\; \mathbf{M}^2\;\left(\;(\frac{dT}{d\omega})^2 \;-\; (\frac{dX}{d\omega})^2 \;+\; \frac{(\frac{dE}{d\omega})^2 \;-\; (\frac{dP}{d\omega})^2}{b^2}\;\right) \;=\; \mathbf{M}^2\;\frac{(d\omega)^2}{(d\omega)^2} \;=\; \mathbf{M}^2 \tag{3.4}$$

recovering now the generalized dispersion relation in flat phase space and which is invariant under velocity and force/acceleration boosts transformations (1.4).

As stated earlier, what is an invariant is the phase space interval

$$b^2\;(d\omega)^2 \;=\; b^2\;\left(\;(dT)^2 \;-\; (dX)^2\;\right) \;+\; (dE)^2 \;-\; (dP)^2 = \; b^2\;(d\tau)^2 \;-\; (d\mathcal{M})^2 \tag{3.5}$$

where the (spacelike) mass infinitesimal displacement is defined by

$$-(d\mathcal{M})^2 \;\equiv\; (dE)^2 \;-\; (dP)^2 \;\leq\; 0 \tag{3.6}$$

given $E^2 - P^2 = m^2$, eq-(3.6) can then be rewritten as

$$-\;(d\mathcal{M})^2 \;=\; -\;\frac{m^2}{P^2 + m^2}\;(dP)^2 \;\leq\; 0 \tag{3.7}$$

When $b \neq \infty$, for massless particles $m = 0$ like the photon, eq- (3.7) leads to the *null* intervals in phase space

$$(d\omega)^2 \ = \ 0, \ \ (d\tau)^2 \ = \ 0, \ \ (d\mathcal{M})^2 \ = \ 0, \ \ (dX)^2 \ = \ (dT)^2, \ \ (dE)^2 \ = \ (dP)^2 \tag{3.8}$$

as expected. When $b = \infty$, we recover the standard relativistic expression $d\omega = d\tau$. Integrating eq-(3.7) yields

$$\mathcal{M} \ = \ m \ ln \left(\frac{\sqrt{P^2 + m^2} + P}{m} \right) \tag{3.9}$$

Let us look at the case of uniform *linear* acceleration. The hyperbolic word-line in Minkowski spacetime of a uniformly accelerated particle (observer) along the X axis, with constant proper acceleration g, is given by

$$T \ = \ \frac{1}{g} \ sinh(g\tau), \ \ X \ = \ \frac{1}{g} \ cosh(g\tau) \tag{3.10}$$

from which one infers that the (spacelike) proper acceleration is

$$(\frac{d^2T}{d\tau^2})^2 \ - \ (\frac{d^2X}{d\tau^2})^2 \ = \ - \, g^2 \tag{3.11}$$

when the signature is chosen to be $(+,-)$. The maximal proper force condition in this case simply amounts to $mg \leq b$. An upper bound on $g \leq \frac{b}{m}$ implies a *minimum* value of X lying *inside* the Rindler wedge $X \geq \frac{m}{b}$. When $b = m^2_{Planck} = (2G)^{-1} \Rightarrow X \geq 2Gm = r_h$. The minimum X distance coincides with the black hole horizon radius r_h associated with a particle of mass m. For a Planck mass particle the minimum X value would be the Planck scale. Whereas in the uniform circular motion, we found the the radius of the circle cannot exceed the value of the horizon radius [17].

The phase space interval is then

$$(d\omega)^2 \ = \ (dT)^2 - (dX)^2 + \frac{(dE)^2 \ - \ (dP)^2}{b^2} \ = \ (d\tau)^2 \ (1 - \frac{F^2}{b^2}) \ = \ (d\tau)^2 \ (1 - \frac{(mg)^2}{b^2}) \ \Rightarrow$$

$$(d\mathcal{M})^2 \ = \ (mg)^2 \ (d\tau)^2 \ \Rightarrow \ \mathcal{M} \ = \ mg\tau \tag{3.12}$$

One can verify after some algebra that eq-(3.12) agrees with eq-(3.9) when

$$P \equiv m \ \gamma(v) \ v, \ \ v \ = \ \frac{dX}{dT} \ = \ tanh(g\tau) \ \Rightarrow$$

$$P \ = \ m \ (1 - tanh^2(g\tau))^{-1/2} \ tanh(g\tau) \ = \ m \ sinh(g\tau) \ \Rightarrow$$

$$\mathcal{M} \ = \ m \ ln \left(\frac{\sqrt{P^2 + m^2} + P}{m} \right) \ = \ m \ ln(cosh(g\tau) + sinh(g\tau)) \ =$$

$$m \ ln(e^{g\tau}) \ = \ mg\tau \tag{3.13}$$

as expected.

To su, up, from eqs-(3.3,3.4) one can explicitly confirm that $\mathcal{P}_M \mathcal{P}^M = \mathbf{M}^2$ is the invariant generalized dispersion relation in phase-space, while the *modified* dispersion relation in *spacetime* is given in this particular case by

$$\mathcal{P}_\mu \mathcal{P}^\mu \; = \; \frac{\mathbf{M}^2}{1 - \frac{F^2}{b^2}} \; = \; \mathbf{M}^2 \; \left(\; 1 \; + \; \frac{F^2}{b^2} \; + \; (\frac{F^2}{b^2})^2 \; + \; \cdots \; \right) \; =$$

$$\mathbf{M}^2 \; + \; \mathbf{M}^2 \; \frac{F^2}{b^2} \; + \; \mathbf{M}^2 \; (\frac{F^2}{b^2})^2 \; + \; \ldots \tag{3.14}$$

This is a clear example of how one can have *modified* dispersion relations in *spacetime* due to the corrections based on the maximal force relativity principle. A salient feature of the spacetime modified dispersion relations (3.14) is that they are still Lorentz invariant. One has been able to write down *modified* dispersion relations without destroying Lorentz invariance. The reason is because the spacetime momentum components of $\mathcal{P}^M$ are now given by $\mathcal{P}^\mu = \mathbf{M}(dX^\mu/d\omega) \neq P^\mu = m(dX^\mu/d\tau)$. The old definition is based on the mass parameter m and in taking derivatives with respect to the proper time τ. As stated earlier the true invariant under velocity and force/acceleration boosts is ω, and by inspection of eqs-(1.8) one can see that $P_\mu P^\mu$, and $\mathcal{P}_\mu \mathcal{P}^\mu$ are *not* invariant under force/acceleration boosts transformations, despite being Lorentz invariant. What is truly invariant is $\mathcal{P}_M \mathcal{P}^M = \mathbf{M}^2 \neq m^2$.

2.2 Generalized Force in Flat Phase Space

The components of the *generalized* proper force in (flat) phase space are now given by

$$\mathcal{F}^M \; \equiv \; \mathbf{M} \; \frac{d^2 Z^M}{d\omega^2} \; = \; \mathbf{M} \; \left(\; \frac{d^2 T}{d\omega^2}, \; \frac{d^2 X}{d\omega^2}, \; \frac{d^2 E}{d\omega^2}, \; \frac{d^2 P}{d\omega^2} \; \right) \tag{3.15}$$

In particular, the X, T-components of the generalized proper force $\mathcal{F}^M$ are given by

$$\mathcal{F}_X \; = \; \frac{\mathbf{M}}{1 - \frac{F^2}{b^2}} \; \frac{d^2 X}{d\tau^2} \; - \; \frac{\mathbf{M}}{b^2} \; \frac{(dX/d\tau) \; F_\mu \; (dF^\mu/d\tau)}{(1 - \frac{F^2}{b^2})^2} \tag{3.16a}$$

$$\mathcal{F}_T \; = \; \frac{\mathbf{M}}{1 - \frac{F^2}{b^2}} \; \frac{d^2 T}{d\tau^2} \; - \; \frac{\mathbf{M}}{b^2} \; \frac{(dT/d\tau) \; F_\mu \; (dF^\mu/d\tau)}{(1 - \frac{F^2}{b^2})^2} \tag{3.16b}$$

$\mathcal{F}_X$ is related to actual physical forces (rate of change of momentum), while $\mathcal{F}_T$ is related to power (rate of change of energy).

When there is a *uniform linear* acceleration in spacetime, the proper force in spacetime is *constant* and spacelike $F_\mu F^\mu = -F^2 \Rightarrow F_\mu (dF^\mu/d\tau) = 0$, thus

$\mathcal{F}_X$ becomes in this special case

$$\mathcal{F}_X \;=\; \frac{\mathbf{M}}{1-\frac{F^2}{b^2}}\;\frac{d^2X}{d\tau^2} \qquad (3.17)$$

when $F^2/b^2 < 1$, a Taylor expansion of eq-(3.17) yields

$$\mathcal{F}_X \;=\; \mathbf{M}\;\left(\;1\;+\;\frac{F^2}{b^2}\;+\;(\frac{F^2}{b^2})^2\;+\;\cdots\;\right)\;\frac{d^2X}{d\tau^2} \qquad (3.18)$$

Consequently, the *first* order corrections ΔF_X to the standard spacetime relativistic proper force $\mathbf{M}(d^2X/d\tau^2)$, due to the maximal proper force principle, are given by

$$\Delta F_X \;=\; \mathbf{M}\;\frac{F^2}{b^2}\;\frac{d^2X}{d\tau^2}\;=\;\mathbf{M}g\;(\frac{mg}{b})^2\;cosh(g\tau)\;=\;\Delta F\;cosh(g\tau)\;>\;0 \quad (3.19)$$

where $\Delta F = \mathbf{M}g\;(\frac{mg}{b})^2$. What is numerically relevant is the relative fractional correction of the force (to first order)

$$\frac{\Delta F_X}{F_X}\;=\;\frac{F^2}{b^2}\;=\;(\frac{mg}{b})^2\;<\;1 \qquad (3.20)$$

In the limiting case $F^2/b^2 = 1$ the generalized force components in eqs-(3.16) blow up $\mathcal{F}_X = \mathcal{F}_T = \infty$. A similar scenario occurs in ordinary Special Relativity, it takes an infinite energy to accelerate a massive particle to the speed of light. A salient feature of eq-(3.18) is that the *first* order (and higher order) corrections $\Delta F_X > 0$ are *repulsive* for all values of τ. This is is reminiscent of the presently observed repulsive force involved in the accelerated expansion of the universe.

Let us evaluate the components of the generalized proper force in phase space (3.15), in the special case when $F = mg = constant$, and associated to a massive particle with a uniform linear acceleration (Rindler observer). In this case, eq-(3.10) (in $c = 1$ units) yield

$$v\;=\;tanh(g\tau),\quad \gamma(v)\;=\;(1-v^2)^{-1/2}\;=\;cosh(g\tau),$$
$$E\;=\;m\gamma(v)\;=\;m\;cosh(g\tau),\quad P\;=\;m\;\gamma(v)\;v\;=\;m\;sinh(g\tau) \qquad (3.21)$$

$$\mathcal{F}_X\;=\;\frac{\mathbf{M}}{1-\frac{F^2}{b^2}}\;\frac{d^2X}{d\tau^2}\;=\;\frac{\mathbf{M}g\;cosh(g\tau)}{1-\frac{F^2}{b^2}} \qquad (3.22a)$$

$$\mathcal{F}_T\;=\;\frac{\mathbf{M}}{1-\frac{F^2}{b^2}}\;\frac{d^2T}{d\tau^2}\;=\;\frac{\mathbf{M}g\;sinh(g\tau)}{1-\frac{F^2}{b^2}} \qquad (3.22b)$$

$$\mathcal{F}_P\;=\;\frac{\mathbf{M}}{1-\frac{F^2}{b^2}}\;\frac{d^2P}{d\tau^2}\;=\;\frac{\mathbf{M}mg^2\;sinh(g\tau)}{1-\frac{F^2}{b^2}} \qquad (3.22c)$$

$$\mathcal{F}_E\;=\;\frac{\mathbf{M}}{1-\frac{F^2}{b^2}}\;\frac{d^2E}{d\tau^2}\;=\;\frac{\mathbf{M}mg^2\;cosh(g\tau)}{1-\frac{F^2}{b^2}} \qquad (3.22d)$$

The norm squared of the generalized proper force in phase space when $F = mg$ is then given by

$$\Omega^2 \equiv \mathcal{F}_M \mathcal{F}^M = (\mathcal{F}_T)^2 - (\mathcal{F}_X)^2 + \frac{(\mathcal{F}_E)^2 - (\mathcal{F}_P)^2}{b^2} =$$

$$- \mathbf{M}^2 \frac{g^2}{1 - \frac{F^2}{b^2}} = - (\mathbf{MA})^2, \quad \mathbf{A} \equiv \frac{g}{\sqrt{1 - \frac{F^2}{b^2}}} \tag{3.23}$$

When $F = mg = b$ reaches the maximal proper force value in spacetime, we find that the norm squared of the *generalized* proper force in *phase space* is $\Omega^2 = -\infty$.

One could write eq-(3.23) in the form

$$\mathbf{M}(F) = \frac{\mathbf{M}}{\sqrt{1 - \frac{F^2}{b^2}}} \leftrightarrow m(v) = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3.24}$$

and $\Omega^2 \equiv -(\mathbf{F})^2$, with

$$\mathbf{F} = \mathbf{M}(F)\; g = \frac{\mathbf{M}g}{\sqrt{1 - \frac{F^2}{b^2}}} \leftrightarrow P = m(v)\; v = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3.25}$$

because it implements Born's Reciprocal Relativity Principle in a natural fashion. On the right hand side we have the velocity dependence of the mass $m(v)$, and the momentum, in terms of the *rest* mass m and the velocity v. In the left hand side we have the force-dependent mass $\mathbf{M}(F)$ which is given in terms of the force $F = mg$, and the inertial mass $\mathbf{M}$ associated to a non-accelerated frame (either at rest or moving with a constant velocity).

Furthermore, given $F = mg \leq b$, one could write $b = mA$, where $A = b/m$ is the maximal acceleration that a particle of mass m can sustain. The more massive the particle is the lower A is, and vice versa. Hence the left hand side eq-(3.25) can be rewritten in terms of A as

$$\mathbf{F} = \mathbf{M}(g)\; g = \frac{\mathbf{M}g}{\sqrt{1 - \frac{g^2}{A^2}}} \leftrightarrow P = m(v)\; v = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3.26}$$

such that the correspondence with the right hand side is more evident.

In the case of a ($\mathbf{M} = 0$) massless particle (photon), the trajectory in phase-space defined by $T = X;\ E = P,\ d\omega = d\tau = 0$, and described in terms of an affine parameter $\lambda = \omega/\mathbf{M}$, yields *identically zero* $\mathcal{F}_M\mathcal{F}^M = \Omega^2 = 0,\ F_\mu F^\mu = F^2 = 0$ values. Hence, a photon describes a *null* path, and it experiences a zero proper force magnitude, both in spacetime *and* in phase-space.

Whereas a massive particle subjected to the maximal proper force b in *spacetime* will experience an *infinite* generalized proper force squared in phase-space $\Omega^2 = -\infty$, while having a *null* interval in phase-space $(d\omega)^2 = (d\tau)^2(1 - F^2/b^2) = 0$, but a *timelike* interval in spacetime $(d\tau)^2 > 0$.

To conclude this section, we must emphasize that in the most general case, when $F_\mu F^\mu = -F^2(\tau) \neq constant$, eqs-(3.16) would lead to a system of coupled nonlinear third order differential equations for $X = X(\tau), T(\tau)$ once $\mathcal{F}_X(\tau), \mathcal{F}_T(\tau)$ are known. To solve them in the most general case would be a very difficult task.

2.3 Cosmological Applications : Dark Matter

From these equations (3.24-3.26) one can conclude that the generalized mass $\mathbf{M}$ corresponding to the motion in flat phase-space (the cotangent bundle of spacetime) is an increasing function of the force (acceleration) experienced by the particle in the underlying spacetime base manifold, and it becomes infinite at the maximal proper force $F = mg = b$. This is similar with what happens in Special Relativity (SR). The mass $m(v) = m\gamma(v) = m(1 - v^2/c^2)^{-1/2}$ blows up when the particle reaches the speed of light.

This might have $profound$ consequences in Cosmology. The dark matter, dark energy puzzle might be related to this effect if we embed the $4D$ spacetime dynamics of the matter in our Universe into its $8D$ cotangent bundle dynamics. In doing so, the matter in the accelerated expanding Universe will experience a (generalized) mass *increase* described by eq-(3.24), which in turn, will have an *increased* gravitational effect on the surrounding matter, and *mimicking* the effects of dark matter.

A simplified calculation can be obtained by studying the transformation properties of the *line* energy densities $\lambda = energy/length$, rather than volume energy densities ρ, under force/acceleration boosts. From eqs-(1.8) one has

$$dE' = cosh\xi \ dE \ - \ b \ sinh\xi \ dX, \quad dX' = cosh\xi \ dX \ - \ sinh\xi \ \frac{dE}{b} \ \Rightarrow$$

$$\lambda' = \frac{dE'}{dX'} = \frac{\frac{dE}{dX} - b \ tanh\xi}{1 - \frac{\lambda \ tanh\xi}{b}} = \frac{\lambda - b \ tanh\xi}{1 - \frac{\lambda \ tanh\xi}{b}} = \frac{\lambda - F}{1 - \frac{\lambda F}{b^2}} \tag{3.27}$$

after using the definition for the force/acceleration boost parameter $tanh\xi = F/b$. One can polish further eq-(3.27) by noticing that λ has units of mass/length, which are the same units as those of a force when $c = 1$. Since b is the maximal proper force, it is also the maximal line density. Hence, we may divide both sides of eq-(3.27) by b and arrive at

$$\frac{\lambda'}{b} = \frac{\frac{\lambda}{b} - \frac{F}{b}}{1 - \frac{\lambda}{b} \ \frac{F}{b}} \tag{3.28}$$

which is very reminiscent of the subtraction law for velocties in Special Relativity if one writes $\frac{F}{b} = \lambda''$. Changing the sign of F will lead to an addition law. The maximum values that $\frac{\lambda}{b}$, and $\frac{F}{b} = \frac{\lambda''}{b}$ can take are 1, and such that the composition rule reveals that the maximum values that $\frac{\lambda'}{b}$ can take is also 1.

Now we may recall that in the standard model of Cosmology the present-day dark matter plus dark energy amounts roughly to 95 percent, and the ordinary matter (plus radiation) amounts to 5 percent. These numbers actually refer to present-day values of the density parameters Ω's which are defined by the *ratios* of the corresponding matter/energy *densities* with respect to the present-day critical energy density $\rho_{crit} \equiv \frac{3}{8\pi G R_H^2}$ that is defined in terms of the present-day value of the Hubble scale R_H. Thus $\Omega \equiv (\rho/\rho_{crit})$, [1] and the total value Ω_T of the sum of the Ω's is 1 by definition. If one includes the spatial curvature density parameter Ω_k associated a Friedman-Lemaitre-Robertson-Walker (FLRW) model, $k = 1, 0, -1$, the total value for Ω_T may differ from unity.

Following the same procedure, we shall define our *line* density parameters $\sigma = \lambda/b$ as the *ratios* of the corresponding *line* energy *densities* λ with respect to the *maximal* line energy density b. In doing so we can recognize that the composition law (3.28) is nothing but the law corresponding to the composition of line density parameters

$$\sigma' \ = \ \frac{\sigma \pm \sigma''}{1 \pm \sigma \ \sigma''}, \quad \sigma \equiv \frac{\lambda}{b}, \ \sigma'' \equiv \frac{F}{b}, \quad \sigma' \equiv \frac{\lambda'}{b} \tag{3.29}$$

In passing, we should mention that the maximal proper force b (maximal line energy density) is in fact related to the observed mass M_U of the Universe within the present-day Hubble scale R_H, and which in turn, leads to the derivation of the critical energy density $\rho_{crit} \equiv \frac{3}{8\pi G R_H^2}$. We shall show this below.

Finally, we can infer the crucial physical significance of the composition law of the line density parameters in eq-(3.29). If the value of the baryonic matter density parameter is $\sigma = 5/100$, for example, its value in an accelerated frame of reference σ' will be *higher* under the addition law (lower under the subtraction law). In particular, under the addition law in eq-(3.29), in the frame of reference when F reaches its maximal value, $\sigma'' = \frac{F}{b} = 1 \Rightarrow \sigma' = \lambda'/b = 1$, thus the line density parameter reaches its *maximum* value of 1. Choosing a lower value for F will yield $\sigma' < 1$, like 95/100, and so forth. Consequently, what we may interpret as dark matter surrounding galaxies, could be just an effect of observing ordinary galactic matter in different accelerating frames of reference than ours. Since the universe's expansion is accelerating, galaxies are clearly accelerating with respect to us as observed by Hubble long ago. In the next section we shall provide numerical examples.

2.4 Maximal Force, Vacuum Energy and Black-Hole Cosmology

It was shown in [18] how one can implement the Maximal Force Relativity principle within the context of Mach's principle and Black-Hole Cosmology [19]

[1]Not to be confused with Ω in eq-(3.23)

by setting the following proper forces to be equal to the maximal proper force value b

$$M_U\ (\frac{c^2}{R_H})\ =\ m_P\ (\frac{c^2}{L_P})\ =\ b \tag{3.30}$$

where M_U is the Universe's total mass inside the present-day Hubble radius R_H; m_P, L_P are the Planck mass, and length, respectively. The Planck length is defined by equating the Schwarzschild radius $2Gm/c^2$ to the Compton wavelength $\hbar/mc$. In doing so one obtains $\hbar/mc = L_P = \sqrt{\frac{2\hbar G}{c^3}}$. In units of $\hbar = c = 1$, eq-(3.30) gives

$$b\ =\ \frac{M_U}{R_H}\ =\ (m_P)^2\ =\ (L_P)^{-2}\ =\ \frac{1}{2G} \tag{3.31}$$

if the mass M_U is distributed uniformly over the volume of a sphere of radius R_H, the energy density is

$$\rho\ =\ \frac{M_U}{(4\pi/3)R_H^3}\ =\ \frac{3}{4\pi}\ \frac{M_U}{R_H}\ \frac{1}{R_H^2}\ =\ \frac{3}{8\pi G R_H^2}\ =\ \rho_{crit} \tag{3.32}$$

which coincides with the present-day critical energy density, the observed vacuum energy density ρ_{vac} driving the exponential expansion of the universe in a late time de Sitter phase. There is yet another interpretation of this result. From eqs-(3.30, 3.31) one learns that the work (energy) performed by the maximal proper force b over a distance R_H is

$$E\ =\ b\ R_H\ =\ \frac{R_H}{2G}\ =\ M_U\ \Rightarrow\ R_H\ =\ 2GM_U \tag{3.33}$$

What these results (3.33) indicate is that the observed Universe's total mass M_U coincides with the product of the maximal proper force times the Hubble horizon scale (an infrared cutoff), and which in turn, is the black hole horizon radius corresponding to a Universe-mass black hole. The Planck mass is the product of the maximal proper force times the Planck scale (ultraviolet cutoff), and which in turn, is the black hole horizon radius corresponding to a Planck-mass black hole. And so forth, namely a black hole's mass M coincides with the product of the maximal proper force b with its black hole horizon radius R.

Let us now provide some results when one looks at the distribution of ordinary matter. To simplify matters considerably in our calculations we may assume a spherical (volume) mass density distribution $\rho(r)$ such that the line matter density is $\lambda(r) = 4\pi\rho(r)r^2$, and is obtained simply by equating the mass stored inside a spherical shell at radius $r, r+dr$: $dm = \lambda(r)dr = 4\pi\rho(r)r^2dr$. Adopting Hubble's expansion law that velocity is proportional to distance $v(r) = H_o r$, where $H_o = 1/R_H$ is the Hubble parameter present-day value, the acceleration is $a(r) = H_o(dr/dt) = H_o v(r) = H_o^2 r$. Hence the force experienced by an infinitesimal mass element dm (inside each expanding spherical shell) is $dF = a(r)(dm) = (H_o^2 r)\ (4\pi\rho(r)r^2dr)$. A word of *caution* : note that rigorously speaking this force dF is *not* a proper-force mg as the one described by eqs-(3.10, 3.11) for the uniform linear accelerated motion (Rindler observers). Nevertheless for the sake of the argument, and to simplify matters, we shall use dF, otherwise we should provide an *ensemble* of Rindler observers, each attached to every expanding spherical shell.

Summarizing one has

$$\lambda(r) = 4\pi\rho(r)r^2 \;\Rightarrow\; d\lambda \;=\; 8\pi\;\rho(r)\;r\;dr \;+\; 4\pi\;r^2\;\frac{d\rho(r)}{dr}\;dr \tag{3.34a}$$

$$dF \;=\; 4\pi\;H_o^2\;\rho(r)\;r^3\;dr \;\Rightarrow\; F(r) - F(r=0) \;=\; 4\pi\;H_o^2\;\int_0^r\;\rho(r)\;r^3\;dr \tag{3.34b}$$

Having at our disposal these equations (3.34) one can then differentiate the line-density addition law (3.28)

$$\lambda_2 \;=\; \frac{\lambda_1 \;+\; \lambda_2}{1 \;+\; \frac{\lambda_1\;\lambda_2}{b^2}} \;\Rightarrow$$

$$d\lambda_2 \;=\; \frac{(1 \;+\; \frac{\lambda_1\;\lambda_2}{b^2})\;(d\lambda_1 \;+\; d\lambda_2) \;-\; (\lambda_1 \;+\; \lambda_2)\;b^{-2}d(\lambda_1\lambda_2)}{(1 \;+\; \frac{\lambda_1\;\lambda_2}{b^2})^2} \;\Rightarrow \tag{3.35}$$

$$\lambda_2(R_H) \;=\; \int_0^{R_H}\;\frac{(1 \;+\; \frac{\lambda_1\;\lambda_2}{b^2})\;(d\lambda_1 \;+\; d\lambda_2) \;-\; (\lambda_1 \;+\; \lambda_2)\;b^{-2}d(\lambda_1\lambda_2)}{(1 \;+\; \frac{\lambda_1\;\lambda_2}{b^2})^2} \tag{3.36}$$

where we chose the boundary condition $\lambda_2(r=0) = 0$. Inserting the following expressions

$$\lambda_1(r) \;=\; 4\pi\rho(r)r^2,\quad \lambda_2(r) \;=\; F(r) - F(r=0) \;=\; 4\pi\;H_o^2\;\int_0^r\;\rho(r)\;r^3\;dr,$$

$$d\lambda_1 \;=\; 8\pi\;\rho(r)\;r\;dr + 4\pi\;r^2\;\frac{d\rho(r)}{dr}\;dr,\quad d\lambda_2 \;=\; dF \;=\; 4\pi\;H_o^2\;\rho(r)\;r^3\;dr \tag{3.37}$$

into the integral (3.36) it will give us the sought-after expression for the *cumulative* and *inflated* line matter density inside the Hubble radius $\lambda_2(r = R_H)$, and given in terms of the spherical volume mass density $\rho(r)$ distribution. In order to evaluate the integral one needs to know the expression for $\rho = \rho(r)$ which must be *compatible* with the addition composition law (3.28) that imposes that $\lambda_2(r)$ cannot exceed the maximal value of b.

The integral in (3.36) can be seen as a functional of $\rho(r) : I[\rho(r)] =$ real number. Hence one can obtain a wide range of numerical values depending on $\rho(r)$. But it *cannot* be an arbitrary function because given that $\lambda(r) = 4\pi\rho(r)r^2$, and that $\lambda(r) \leq b$ *cannot* exceed the maximal density b, one can infer from eqs-(3.30-3.32) that the volume mass density $\rho(r)$ must be constrained to obey the following conditions

$$\rho(r) \; < \; \rho_{vac} \; (\frac{R_H}{r})^2 \; = \; \rho_{Planck} \; (\frac{L_{Planck}}{r})^2 \qquad (3.37)$$

$$\rho_{vac} \; = \; \frac{3}{8\pi G R_H^2}, \quad \rho_{Planck} \; = \; \frac{3 m_{Planck}}{4\pi L^3_{Planck}} \qquad (3.38)$$

If $\rho(r) = \rho_o = constant$, the condition (3.37) would require us to introduce an infrared-cutoff $r_2 = R_H - \epsilon$ to force $\rho_o < \rho_{vac}$. This is a reasonable choice because in this case the value of ρ_o would be very close to the ρ_{vac} when ϵ is very small.

The boundary condition $\lambda(r = 0) = 0$ requires that in the regions very near the origin $r = 0$ the density behaves $\rho(r) \simeq r^{\delta - 2}$, for $\delta > 0$. If $2 > \delta > 0 \Rightarrow \rho(r = 0) = \infty$ which is unphysical and would force us to introduce an ultraviolet cut-off, say the Planck scale, to avoid this singularity. If $\delta > 2 > 0$, then $\rho(r = 0) = 0$ which is well behaved. Having a zero density at $r = 0$ is not farfetched since there are vast empty regions devoid of matter in the Universe.

Concluding, in this very specific and simplified model, after obtaining a numerical value for the integral $I[\rho(r)]$ defining λ_2, and upon dividing it by b, one will then find the *cumulative inflated* percentage ratio of the matter content of the Universe due to the maximal-force relativistic corrections based on Born's reciprocity principle, and originating from the *accelerated* expansion of the Universe. Let us imagine that for a judicious choice of $\rho(r)$ one finds, after evaluating the integral (3.36) that the ratio becomes $\lambda_2/b = 30/100$, when $\lambda_1/b = 5/100$. This would entail that the inflated ratio, due to the accelerated expansion of the Universe, would amount to an increase of 25/100 and, consequently, one might be inclined to conclude that there is "dark matter" out there in the Universe which should account for this 25/100 increase.

Some readers may argue that these results are just mere numerical coincidences arising in Black-Hole Cosmology [19], [22]. We beg to differ and postulate that the underlying physical origins behind these numerical coincidences may stem from the maximal proper-force relativity theory based on Born's reciprocity principle. Another interpretation of eqs-(3.30-3.33) [24] is that involving matter creation from the vacuum, as advocated by Hoyle long ago. Imagine one pumps matter out of the vacuum in lumps/units of Planck masses. Let us assume that the Universe expands in such a way that matter is being replenish from the vacuum so that the mass at any moment is linearly proportional to the size of the Universe. As the mass of the universe grows the vacuum energy density decreases since the vacuum is being depleted. In this scenario, at the Hubble scale R_H, one has $M_U \sim R_H$.

Seven-year Wilkinson Microwave Anisotropy Probe (WMAP) observations show that our Universe may be indeed closed, with the total density parameter $\Omega = 1.0023$ [20]. The critical density ρ_{crit} is the total density of matter/energy needed for the universe to be spatially flat. Measurements indicate that the actual total density is very close if not equal to this value.

The author [21] has proposed that the closed Universe is the interior of a black hole existing in another universe. Quantum particle production caused by an extremely high curvature near a bounce creates enormous amounts of matter, produces entropy, and generates a period of exponential expansion (inflation) of this universe. Depending on the particle production rate, such a universe may undergo several nonsingular bounces until it has enough matter to reach a size at which the cosmological constant starts cosmic acceleration. The last bounce can be regarded as the big bang (or rather the big bounce) of this universe.

However, the nature of dark matter and dark energy still remains unsolved in this scenario [21]. A more recent proposal that advocates the fall of dark matter may be found in [23]. Scale invariance is assumed in the empty regions of space. The Weyl gauge field A_μ contributes to modifications of the Christoffel connection leading then to repulsive corrections to the geodesic equations. Another approach to solve these cosmological puzzles based on the scaling properties of fractals can be found in Nottale's Scale Relativty Theory [25]. A more recent approach to derive the value of ρ_{vac} based on the novel Bohm-Poisson-Schroedinger equation, and which also explains the repulsive force of dark energy, can be found in [24]. For an extensive review of the successes and problems of the Standard Models of Cosmology we refer to [26].

3 Finsler Geometry and Curved Phase Space

To study the geometry behind a maximal proper force and/or maximal acceleration in more general *curved* phase spaces (cotangent bundles), we shall follow next the description by the authors [16] where one may study in detail the geometry of Lagrange-Finsler and Hamilton-Cartan spaces and their higher order (jet bundles) generalizations.

In the case of the cotangent space of a d-dim manifold T^*M_d the metric can be equivalently rewritten in the block diagonal form as

$$(d\omega)^2 \ = \ g_{ij}(x^k, p_a)\ dx^i d\ x^j \ + \ h^{ab}(x^k, p_c)\ \delta p_a\ \delta p_b \ = $$
$$g_{ij}(x^k, p_a)\ dx^i d\ x^j \ + \ h_{ab}(x^k, p_c)\ \delta p^a\ \delta p^b \tag{4.1}$$

$i, j, k = 1, 2, 3, ...d$, $a, b, c = 1, 2, 3, ...d$, if instead of the standard coordinate basis one introduces the following anholonomic frames (non-coordinate basis)

$$\delta_i \ = \ \delta / \delta x^i \ = \ \partial_{x^i} \ + \ N_{ia}\ \partial^a \ = \ \partial_{x^i} \ + \ N_{ia}\ \partial_{p_a}; \ \ \partial^a \ \equiv \ \partial_{p_a} \ = \ \frac{\partial}{\partial p_a} \tag{4.2}$$

One should note the *key* position of the indices that allows us to distinguish between derivatives with respect to x^i and those with respect to p_a. The dual basis of $(\delta_i = \delta/\delta x^i; \partial^a = \partial/\partial p_a)$ is

$$dx^i, \ \ \delta p_a \ = \ dp_a \ - \ N_{ja}\ dx^j, \ \ \delta p^a \ = \ dp^a \ - \ N^a_j\ dx^j \tag{4.3}$$

where the N–coefficients define a nonlinear connection, N–connection structure.

An N-linear connection D on T^*M can be uniquely represented in the adapted basis in the following form

$$D_{\delta_j}(\delta_i) \;=\; H^k_{ij}\;\delta_k;\;\; D_{\delta_j}(\partial^a) \;=\; -\; H^a_{bj}\;\partial^b; \qquad (4.4a)$$

$$D_{\partial^a}(\delta_i) \;=\; C^{ka}_i\;\delta_k;\;\; D_{\partial^a}(\partial^b) \;=\; -\; C^{ba}_c\;\partial^c \qquad (4.4b)$$

where $H^k_{ij}(x,p), H^a_{bj}(x,p), C^{ka}_i(x,p), C^{ba}_c(x,p)$ are the connection coefficients. Our notation for the derivatives is

$$\partial^a = \partial/\partial p_a, \;\; \partial_i \;=\; \partial_{x^i}, \;\; \delta_i \;=\; \delta/\delta x^i \;=\; \partial_{x^i} \;+\; N_{ia}\;\partial^a \qquad (4.4c)$$

The N–connection structures can be naturally defined on (pseudo) Riemannian spacetimes and one can relate them with some anholonomic frame fields (vielbeins) satisfying the relations $\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = W^\gamma_{\alpha\beta}\delta_\gamma$. The only nontrivial (nonvanishing) nonholonomy coefficients are

$$W_{ija} \;=\; R_{ija};\;\; W^a_{\ jb} \;=\; \partial^a N_{jb} \;=\; -\; W_j{}^a{}_b \qquad (4.5a)$$

and

$$R_{ija} \;=\; \delta_j N_{ia} \;-\; \delta_i N_{ja} \qquad (4.5b)$$

is the nonlinear connection curvature (N–curvature).

Imposing a zero nonmetricity condition of $g_{ij}(x,p), h^{ab}(x,p)$ along the horizontal and vertical directions, respectively, gives

$$D_i g_{jk} \;=\; \delta_i\;g_{gk}\;-H^l_{ij}\;g_{lk}\;-H^l_{ik}\;g_{jl} \;=\; 0, \qquad (4.6a)$$

$$D^a h^{bc} \;=\; \partial^a\;h^{bc} \;+\; C^{ab}_d\;h^{dc} \;+\; C^{ac}_d\;h^{bd} \;=\; 0 \qquad (4.6b)$$

Performig a cyclic permutation of the indices in eqs-(2.6), followed by linear combination of the equations obtained yields the irreducible (horizontal, vertical) h-v-components for the connection coefficients

$$H^i{}_{jk} \;=\; \frac{1}{2} g^{in}\,(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}) \qquad (4.7)$$

$$C^{ab}_c \;=\; -\;\frac{1}{2}\;h_{cd}\;\left(\;\partial^b h^{ad} \;+\; \partial^a h^{bd} \;-\; \partial^d h^{ab}\;\right) \qquad (4.8)$$

The additional conditions $D_i h^{ab} = 0, D^a g_{ij} = 0$, yield the *mixed* components of the connection coefficients

$$H^a_{bj} \;=\; \frac{1}{2}\;\left(\;h^{ac}\;\delta_j h_{bc} \;-\; h^{ac}\;h_{bd}\;\partial^d N_{jc} \;+\; \partial^a N_{jb}\;\right) \qquad (4.9)$$

(after using $h^{ac}\;\delta_j h_{bc} = -(\delta_j h^{ac})\;h_{bc}$)

$$C^{ja}_i \;=\; \frac{1}{2}\;g^{jk}\;\partial^a g_{ik} \qquad (4.10)$$

For any N-linear connection D with the above coefficients the torsion 2-forms are

$$\Omega^i = \frac{1}{2}T^i_{jk}\ dx^j \wedge dx^k + C^{ia}_j\ dx^j \wedge \delta p_a \tag{4.11a}$$

$$\Omega_a = \frac{1}{2}R_{jka}\ dx^j \wedge dx^k + P^b_{aj}\ dx^j \wedge \delta p_b + \frac{1}{2}S^{bc}_a\ \delta p_b \wedge \delta p_c \tag{4.11b}$$

and the curvature 2-forms are

$$\Omega^i_j = \frac{1}{2}R^i_{jkm}\ dx^k \wedge dx^m + P^{ia}_{jk}\ dx^k \wedge \delta p_a + \frac{1}{2}S^{iab}_j\ \delta p_a \wedge \delta p_b \tag{4.12}$$

$$\Omega^a_b = \frac{1}{2}R^a_{bkm}\ dx^k \wedge dx^m + P^{ac}_{bk}\ dx^k \wedge \delta p_c + \frac{1}{2}S^{acd}_b\ \delta p_c \wedge \delta p_d \tag{4.13}$$

where one must recall that the dual basis of $\delta_i = \delta/\delta x^i,\ \partial^a = \partial/\partial p_a$ is given by $dx^i,\ \delta p_a = dp_a - N_{ja}dx^j$.

The distinguished torsion tensors are

$$T^i_{jk} = H^i_{jk} - H^i_{kj};\ S^{ab}_c = C^{ab}_c - C^{ba}_c;\ T^{ia}_j = C^{ia}_j = - T^{ia}{}_j$$

$$P_b{}^a{}_j = H^a_{bj} - \partial^a N_{jb},\ P_b{}^a{}_j = - P_{bj}{}^a$$

$$R_{ija} = \frac{\delta N_{ja}}{\delta x^i} - \frac{\delta N_{ia}}{\delta x^j} \tag{4.14}$$

The distinguished tensors of the curvature are

$$R^i_{kjh} = \delta_h H^i_{kj} - \delta_j H^i_{kh} + H^l_{kj}\ H^i_{lh} - H^l_{kh}\ H^i_{lj} - C^{ia}_k\ R_{jha} \tag{4.15}$$

$$P^{ab}_{cj} = \partial^a\ H^b_{cj} + C^{ad}_c\ P^b_{dj} - \left(\delta_j\ C^{ab}_c + H^b_{dj}\ C^{da}_c + H^a_{dj}\ C^{bd}_c - H^d_{cj}\ C^{ab}_d\right) \tag{4.16}$$

$$P^{ak}_{ij} = \partial^a\ H^k_{ij} + C^{al}_i\ T^k_{lj} - \left(\delta_j\ C^{ak}_i + H^a_{bj}\ C^{bk}_i + H^k_{lj}\ C^{al}_i - H^l_{ij}\ C^{ak}_l\right) \tag{4.17}$$

$$S^{abc}_d = \partial^c\ C^{ab}_d - \partial^b\ C^{ac}_d + C^{eb}_d\ C^{ac}_e - C^{ec}_d\ C^{ab}_e; \tag{4.18}$$

$$S^{ibc}_j = \partial^c C^{bi}_j - \partial^b C^{ci}_j + C^{bh}_j\ C^{ci}_h - C^{ch}_j\ C^{bi}_h \tag{4.19}$$

$$R^a_{bjk} = \delta_k H^a_{bj} - \delta_j H^a_{bk} + H^c_{bj}\ H^a_{ck} - H^c_{bk}\ H^a_{cj} - C^{ca}_b\ R_{jkc} \tag{4.20}$$

Let us go back and write down the $8D$ cotangent space (phase-space) infinitesimal interval

$$(d\omega)^2 \;=\; g_{ij}(x,p)\,dx^i\,dx^j + h_{ab}(x,p)\,(dp^a - N^a_i(x,p)\,dx^i)\,(dp^b - N^b_j(x,p)\,dx^j) \qquad (4.21)$$

where N^a_i is the nonlinear connection

Given a flat cotangent bundle with $g_{ij} = \eta_{ij}; h_{ab} = \frac{\eta_{ab}}{b^2}$ and $N^a_i = 0$ the interval (4.21) reduces to the Born-Green line interval (1.1).

In the very particular case when

$$g_{ij}(x,p) \;=\; g_{ij}(x), \quad h_{ab}(x,p) \;=\; \frac{g_{ab}(x)}{b^2}, \quad N^a_i(x,p) = \Gamma^a_{id}(x)p^d \qquad (4.22)$$

one can identify the spacetime proper time parameter τ with $s = \int (g_{ij}(x)dx^i dx^j)^{1/2}$, and the $8D$ cotangent bundle infinitesimal interval (4.21) becomes

$$(d\omega)^2 \;=\; g_{ij}(x)\;dx^i\;dx^j \;+ \frac{g_{ab}(x)}{b^2}\;(dp^a \;-\; \Gamma^a_{ci}(x)\;p^c\;dx^i)\;(dp^b \;-\; \Gamma^b_{cj}(x)\;p^c\;dx^j) \qquad (4.23)$$

After factoring out the term $(ds)^2 = g_{ij}(x)\;dx^i\;dx^j$, and writing $p^i = m\frac{dx^i}{ds}$, it gives for the right hand side of eq-(4.23)

$$(ds)^2\;\left(\;1\;+\frac{g_{ab}}{b^2}\;(\frac{dp^a}{ds}\;-\frac{1}{m}\;\Gamma^a_{ci}(x)\;p^c\;p^i)\;(\frac{dp^b}{ds}\;-\frac{1}{m}\;\Gamma^b_{cj}(x)\;p^c\;p^j)\right) \qquad (4.24a)$$

A relabeling of indices allows to rewrite $\Gamma^a_{ci}(x)\;p^c\;p^i = \Gamma^a_{cd}(x)\;p^c\;p^d$, giving for the line interval the following expression

$$(d\omega)^2 \;=\; (ds)^2\;\left(\;1\;+\;\frac{F^2(s)}{b^2}\right)\;=\;(ds)^2\;\left(\;1\;-\;\frac{m^2 g^2(s)}{b^2}\right) \qquad (4.24b)$$

and which has been rewritten in terms of the *spacelike* proper-force squared $F^2(s) = -m^2 g^2(s) < 0$, as follows

$$F^2(s) \;=\; g_{ab}(x)\;F_a\;F_b\;\equiv\;g_{ab}(x)\;\frac{D^2x^a}{Ds^2}\;\frac{D^2x^b}{Ds^2}\;\equiv$$

$$g_{ab}(x)\;\left(m\frac{d^2x^a}{ds^2}\;-\frac{1}{m}\;\Gamma^a_{cd}(x)\;m\frac{dx^c}{ds}\;m\frac{dx^d}{ds}\right)\;\left(m\frac{d^2x^b}{ds^2}\;-\frac{1}{m}\;\Gamma^b_{cd}(x)\;m\frac{dx^c}{ds}\;m\frac{dx^d}{ds}\right) \qquad (4.25)$$

To sum up [2], in this special case the $8D$ cotangent bundle infinitesimal interval can be written in terms of the *covariant* proper force $F^a = m\frac{D^2x^a}{Ds^2}$ of a particle moving in an underlying spacetime of metric $g_{ab}(x)$ as displayed in eqs-(4.25) as follows

[2]One could have relabeled the $a, b, c, d\cdots$ indices as $i, j, k, l\cdots$ instead.

$$(d\omega)^2 \ = \ (ds)^2 \ \left(1 \ - \ \frac{m^2 g^2(s)}{b^2} \right) \ \geq \ 0 \tag{4.26}$$

and it yields a bound on the magnitude of the proper-force squared $\frac{m^2 g^2(s)}{b^2} \leq 1$ similar to the bound on the velocity in Special Relativity $v \leq c$ resulting from the condition $(ds)^2 = (dt)^2 (1 - \frac{v}{c}^2) \geq 0$ excluding tachyons.

3.1 Hamilton Spaces

A typical example of these Finsler geometrical structures is Hamilton spaces. With the help of the spacetime metric, its inverse, and the four-momentum p_a of the particle, the dispersion relation can be written covariantly in terms of the Hamiltonian $H(x,p) \ = \ g^{ab}(x) \ p_a \ p_b \ = \ m^2$. One example of Hamiltonians are homogeneous Hamiltonians of the form [27]

$$H(x,p) \ = \ G^{a_1 a_2 \cdots a_n}(x) \ p_{a_1} \ p_{a_2} \ldots p_{a_n} \tag{4.27}$$

The Hamiltonian encodes the dynamics of point particles via the Hamilton equations of motion. These equations determine the trajectory of a point particle in phase space, i.e. in the cotangent bundle T^*M of the spacetime manifold M.

The Hamilton metric g of H is non-degenerate, nearly everywhere on T^*M (excluding the origin) and defined by

$$g_{ab}(x,p) \ \equiv \ \frac{1}{2} \frac{\partial}{\partial p_a} \frac{\partial}{\partial p_b} \ H(x,p) \tag{4.28}$$

The connection coefficients $N_{ab}(x,p)$ of the Hamilton *nonlinear* connection are given by

$$N_{ab}(x,p) \ = \ \frac{1}{4} \left(\{g_{ab}, H\} \ + \ g_{ai} \frac{\partial}{\partial x^b} \frac{\partial}{\partial p_i} H \ + \ g_{bi} \frac{\partial}{\partial x^a} \frac{\partial}{\partial p_i} \ H \right) \tag{4.29}$$

where $\{A(x,p), B(x,p)\}$ are the Poisson brackets of two functions $A(x,p), B(x,p)$ on T^*M

$$\{A(x,p), B(x,p)\} \ = \ \frac{\partial A}{\partial x^a} \frac{\partial B}{\partial p_a} \ - \ \frac{\partial B}{\partial x^a} \frac{\partial A}{\partial p_a} \tag{4.30}$$

This connection is called non-linear since it may depend non-linearly on the momenta. In terms of these connection coefficients we can define a covariant derivative on the cotangent bundle for so-called d-tensors (distinguished tensors). The components of the d-tensor field depend on positions and momenta, not only on positions. The Hamilton non-linear connection coefficients are the unique connection coefficients which satisfy $N_{ab} = N_{ba}$ and $\nabla g_{ab} = 0$.

The Hamilton geometry of the phase space of particles is the one whose motion is characterised by general dispersion relations. In this framework spacetime and momentum space are naturally curved and intertwined, allowing for a simultaneous description of both spacetime curvature and non-trivial momentum space geometry. The authors [27] considered as explicit examples two models for Planck-scale modified dispersion relations, inspired from the q-de Sitter and κ-Poincare quantum groups. In the first case they found the expressions for the momentum and position dependent curvature of spacetime and momentum space, while for the second case the manifold is flat and only the momentum space possesses a nonzero, momentum dependent curvature. In contrast, for a dispersion relation that is induced by a spacetime metric, as in General Relativity, the Hamilton geometry yields a flat momentum space and the usual curved spacetime geometry with only position dependent geometric objects.

Therefore, this picture based on Hamilton geometry is closer to our work presented here, and should be contrasted with the one based on Hopf algebras and noncommutative geometry of the underlying spacetime discussed earlier. We finalize by saying that the generalized vacuum gravitational field equations in curved (co) tangent spaces have been studied by Vacaru [16]. They have the form

$$R_{ij}(x,p) \ - \frac{1}{2} \ g_{ij}(x,p) \ (R+S)(x,p) \ = \ 0,$$

$$S_{ab}(x,p) \ - \frac{1}{2} \ h_{ab}(x,p) \ (R+S)(x,p) \ = \ 0 \qquad (4.30)$$

we argued in [17] that the term $g_{ij}S$ due to the curvature scalar S in momentum space can be interpreted as an effective stress energy tensor in the base spacetime manifold when $g_{ij}S$ only depend on x. While by Born's reciprocity, the term $h_{ab}R$ due to the scalar curvature in spacetime can be interpreted as an effective stress energy tensor in the momentum space when $h_{ab}R$ only depend on p. Solutions, in very special cases, were found by [16]. To find solutions in more general cases is a daunting task.

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