Calculating the Angle between Projections of Vectors via Geometric (Clifford) Algebra

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James Smith

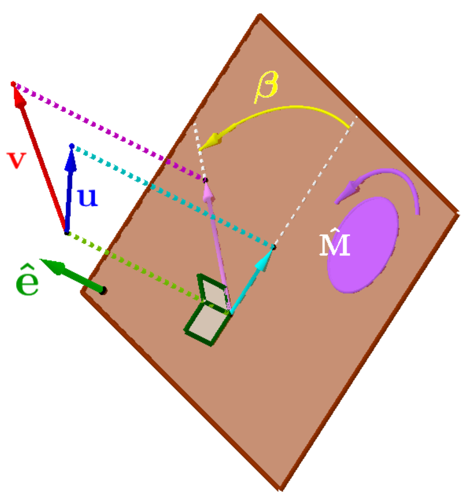
nitac14b@yahoo.com

https://mx.linkedin.com/in/james-smith-1b195047

Abstract

We express a problem from visual astronomy in terms of Geometric (Clifford) Algebra, then solve the problem by deriving expressions for the sine and cosine of the angle between projections of two vectors upon a plane. Geometric Algebra enables us to do so without deriving expressions for the projections themselves.

"Derive expressions for the sine and cosine of the angle of rotation, β, from the projection of $\mathbf{u}$ upon the bivector $\hat{\mathbf{M}}$ to the projection of $\mathbf{v}$ upon $\hat{\mathbf{M}}$."



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1 Introduction

In this document, we will solve—numerically as well as symbolically—a problem of a type that can take the following concrete form, with reference to Fig.1:

> "At a certain location on the Earth, a vertical pole casts a shadow on a perfectly flat, horizontal plaza. On that plaza, local residents

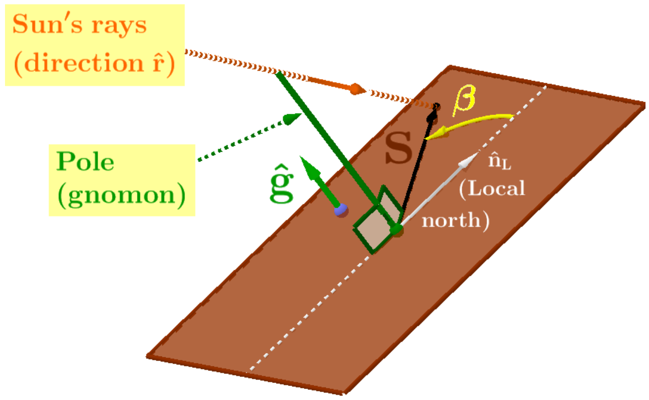

Figure 1: A gnomon (vertical pole) casts a shadow on a perfectly flat, level plaza. The direction of vector $\mathbf{s}$ is from the base of the gnomon to the tip of the shadow. The direction of the Sun's rays is $\hat{\mathbf{r}}$. Upon the plaza, local inhabitants have drawn a line pointing in the direction "local north" ($\hat{\mathbf{n}}_L$). The unit vector in the direction of the plane's normal is $\hat{\mathbf{g}}$.

> who are fans of naked-eye astronomy have traced a north-south line running through the base of the pole. With respect to a right-handed orthonormal reference frame with basis vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, the direction of the Sun's rays is given by the unit vector $\hat{\mathbf{r}} = \hat{\mathbf{a}}r_a + \hat{\mathbf{b}}r_b + \hat{\mathbf{c}}r_c$. The direction of the upward-pointing vector normal to the plaza is $\hat{\mathbf{g}} = \hat{\mathbf{a}}g_a + \hat{\mathbf{b}}g_b + \hat{\mathbf{c}}g_c$, and the direction of the Earth's rotational axis (i.e., the direction from the center of the Earth to the North Pole) is $\hat{\mathbf{n}}_L = \hat{\mathbf{a}}n_{La} + \hat{\mathbf{b}}n_{Lb} + \hat{\mathbf{c}}n_{Lc}$. What is the angle, β, between the north-south line and the pole's shadow?"

2 Formulating the Problem in Geometric-Algebra (GA) Terms, and Devising a Solution Strategy

2.1 Initial Observations

Let's begin by making a few observations that might be useful:

1. A vertical pole that is used to cast a shadow on a flat surface for the purpose of astronomical observations is known as a *gnomon*. We'll use that term in the rest of this document.
2. By saying "the direction of the Sun's rays is $\hat{\mathbf{r}} = \hat{\mathbf{a}}r_a + \hat{\mathbf{b}}r_b + \hat{\mathbf{c}}r_c$", we assumed that all of the Sun's rays are parallel. We'll use that assumption throughout this document.
3. For our purposes, the Earth can be assumed perfectly spherical.

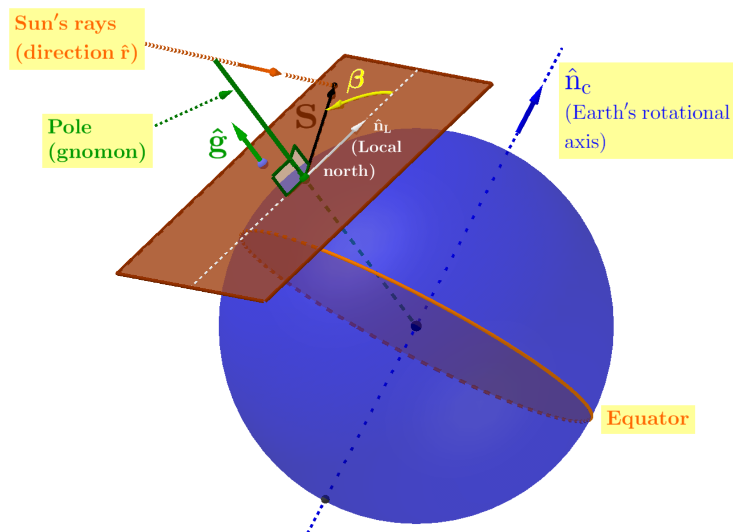

Figure 2: The plaza is a plane tangent to the Earth (assumed spherical) at the point at which the gnomon is embedded .

4. The plaza is a plane tangent to the Earth at the point at which the gnomon is embedded (Fig. 2).

5. The direction of the shadow is the direction of the perpendicular projection of $\hat{\mathbf{r}}$ upon the plaza. Fig. 2 shows why: the gnomon is perpendicular to the plaza, so the shadow is the perpendicular projection some vector $\lambda\hat{\mathbf{r}}$ upon the plaza. Thus, the direction from the base of the gnomon to the tip of the shadow is the same as the direction of the projection of $\hat{\mathbf{r}}$ upon the plaza.

6. The direction from south to north, as traced on the plaza by local residents, is the perpendicular projection of $\hat{\mathbf{n}}_c$ upon the plaza. As proof of that assertion, consider Fig. 3. The south-north line on the plaza is tangent to the great circle that passes through the Earth's North Geographic Pole, and that also passes through the base of the gnomon. The plane that contains that great circle also contains the both $\hat{\mathbf{g}}$ and $\hat{\mathbf{n}}_c$. Therefore, that plane is perpendicular to the plaza. Putting all of these ideas together, the south-north line on the plaza is the projection of some scalar multiple $\mu\hat{\mathbf{n}}_c$ of the vector $\hat{\mathbf{n}}_c$. Thus, that line has the direction of $\hat{\mathbf{n}}_c$'s projection on the plaza.

2.2 Recalling What We've Learned from Solving Similar Problems via GA

Let's refresh our memory about techniques that we may used to solve other problems via GA:

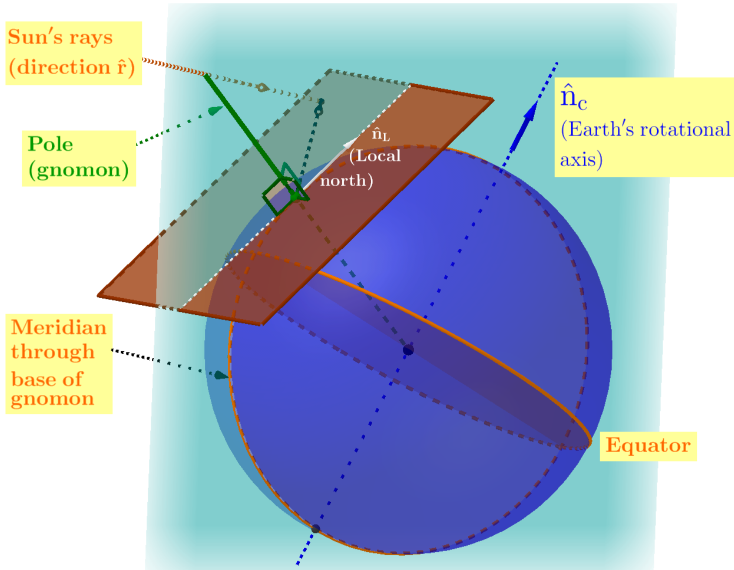

Figure 3: The direction "local north" (i.e., the direction from south to north, as traced on the plaza by local residents) is the same as that of the perpendicular projection of $\hat{\mathbf{n}}_c$ upon the plaza.

1. Problems involving projections onto a plane are usually solved by using the appropriately-oriented bivector that is parallel to the plane, rather than by using the vector that is perpendicular to it. The Appendix (Section 6) shows how to find the required bivector, given said vector. The conclusion is that for the unit perpendicular vector $\hat{\mathbf{e}} = \hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c$, the appropriately-oriented unit bivector is

$$\hat{\mathbf{M}} = \hat{\mathbf{a}}\hat{\mathbf{b}}e_c + \hat{\mathbf{b}}\hat{\mathbf{c}}e_a - \hat{\mathbf{a}}\hat{\mathbf{c}}e_b \tag{2.1}$$

   In GA terms, $\hat{\mathbf{e}}$ is the "dual" of the bivector $\hat{\mathbf{M}}$. We can also see that if we write $\hat{\mathbf{M}}$ in the form $\hat{\mathbf{M}} = \hat{\mathbf{a}}\hat{\mathbf{b}}m_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}m_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}m_{ac}$, then

$$m_{ab} = e_c, \quad m_{bc} = e_a, \quad m_{ac} = -e_b. \tag{2.2}$$

2. The perpendicular projection of a given vector $\mathbf{w}$ upon a given unit bivector, $\hat{\mathbf{M}}$, is (Reference [1], p. 65 , and Ref. [2], p.119):

$$P_{\hat{\mathbf{M}}}(\mathbf{w}) = \left(\mathbf{w} \cdot \hat{\mathbf{M}}\right)\hat{\mathbf{M}}^{-1}. \tag{2.3}$$

   Notation: $P_{\mathbf{C}}(\mathbf{d})$ is the projection of $\mathbf{d}$ upon $\mathbf{C}$.

3. The inverse $\left(\hat{\mathbf{M}}^{-1}\right)$ of the unit bivector $\hat{\mathbf{M}}$ is $-\hat{\mathbf{M}}$.

4. For any vector $\mathbf{y}$ that is parallel to the bivector $\mathbf{A}$, $\mathbf{A}\mathbf{y} = -\mathbf{y}\mathbf{A}$.

5. Putting the last three observations, we arrive at

$$P_{\hat{\mathbf{M}}}(\mathbf{w}) = \hat{\mathbf{M}}\left(\mathbf{w} \cdot \hat{\mathbf{M}}\right). \tag{2.4}$$

6. If two vectors $\mathbf{p}$ and $\mathbf{q}$ are parallel to the bivector $\hat{\mathbf{M}}$, then $\hat{\mathbf{p}}\hat{\mathbf{q}} = e^{\hat{\mathbf{M}}\phi}$, where the scalar ϕ is the angle of rotation from $\mathbf{p}$ to $\mathbf{q}$. Therefore, $\mathbf{p}\cdot\mathbf{q}+\mathbf{p}\wedge\mathbf{q} = \cos\phi + \hat{\mathbf{M}}\sin\phi$. The algebraic sign of ϕ is positive if the direction of that rotation is the same as the orientation of $\hat{\mathbf{M}}$, and negative if in the opposite direction.

7. Equating terms of the same grade in the previous item, we find that

$$\begin{aligned} \cos\phi &= \hat{\mathbf{p}}\cdot\hat{\mathbf{q}}; \\ \sin\phi &= \hat{\mathbf{M}}^{-1}\left(\hat{\mathbf{p}}\wedge\hat{\mathbf{q}}\right). \end{aligned} \tag{2.5}$$

8. Macdonald's definitions (Ref. [2], p. 101) of the inner and outer products are often useful. Those definitions are, for a multivector A of grade j and a multivector B of grade k:

$$\begin{aligned} A_j\cdot B_k &= \langle AB\rangle_{k-j} \text{ (Note: } A_j\cdot B_k \text{ does not exist if } j>k\text{);} \\ A_j\wedge B_k &= \langle AB\rangle_{k+j}. \end{aligned} \tag{2.6}$$

2.3 Further Observations, and Identifying a Strategy

We 've been discussion how to find the angle between the south-north line and the gnomon's shadow. Now, to provide results that will be more generally useful, we'll treat two arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$ (not necessarily unit vectors) and an arbitrary unit bivector, $\hat{\mathbf{M}}$ (Fig. 4). We wish to find the sine and cosine of β, the angle of rotation from $\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})$ to $\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})$.

We could solve the problem by calculating each of those projections according to Eq. (2.3), then calculating the sine and cosine of the requested angle from Eq. (2.5). However, our review of GA in the previous section suggests a strategy that will save us considerable trouble. We'll begin by using Eq. (2.3) to express the two projections that interest us:

$$\begin{aligned} \mathsf{P}_{\hat{\mathbf{M}}}(\hat{\mathbf{u}}) &= \left(\hat{\mathbf{u}}\cdot\hat{\mathbf{M}}\right)\hat{\mathbf{M}}^{-1}; \\ \mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{v}) &= \left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\hat{\mathbf{M}}^{-1}. \end{aligned}$$

Can we now use Eq. (2.5) to calculate $\cos\beta$ and $\sin\beta$? Not yet: although $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, their projects upon $\hat{\mathbf{M}}$ may not be. Therefore, we'll need to calculate $\|\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})\|$ and $\|\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{v})\|$, a detail to which we'll return momentarily. First, we need to calculate $\left[\left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\hat{\mathbf{M}}^{-1}\right]\cdot\left[\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\hat{\mathbf{M}}^{-1}\right]$ and $\left[\left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\hat{\mathbf{M}}^{-1}\right]\wedge\left[\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\hat{\mathbf{M}}^{-1}\right]$.

The definitions of the inner and outer products in Eq. (2.6) use the product A_jB_k, which in our case (because we want to know the rotation from $\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})$ to $\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})$) is

$$\left[\left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\hat{\mathbf{M}}^{-1}\right]\left[\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\hat{\mathbf{M}}^{-1}\right].$$

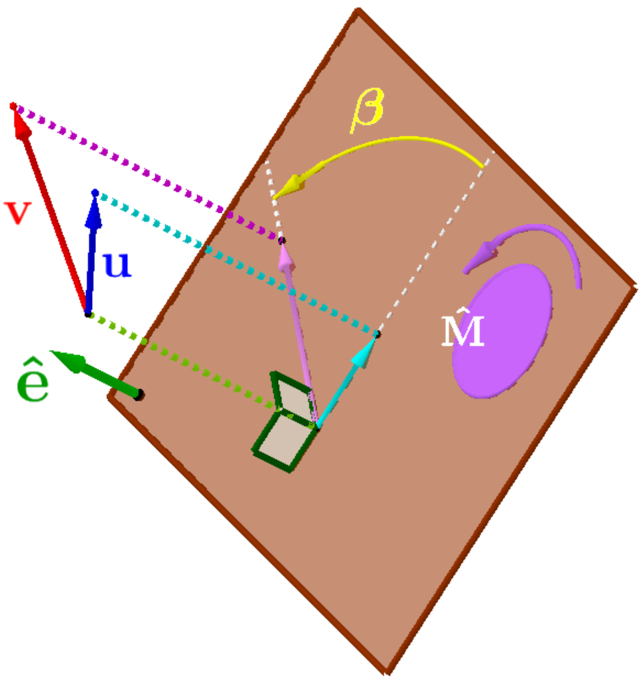

Figure 4: Diagram of the translation, into GA terms, of the more-general type of problem that was motivated by our consideration of the specific case of shadows cast upon a flat, horizontal plaza by a vertical pole. We'll derive expressions for the sine and cosine of the angle of rotation β from the projection of $\mathbf{u}$ upon the bivector $\hat{\mathbf{M}}$ to the projection of $\mathbf{v}$ upon $\hat{\mathbf{M}}$. The vector $\hat{\mathbf{e}}$ is the dual of $\hat{\mathbf{M}}$, and is therefore normal to $\hat{\mathbf{M}}$.

Next, we recognize that $\mathbf{v} \cdot \hat{\mathbf{M}}$ is a vector, and is parallel to $\hat{\mathbf{M}}$. Therefore, using Eq. (2.4), we can write $\left(\mathbf{v} \cdot \hat{\mathbf{M}}\right) \hat{\mathbf{M}}^{-1}$ as $\hat{\mathbf{M}} \left(\mathbf{v} \cdot \hat{\mathbf{M}}\right)$. Using this idea in the previous equation,

$$\begin{aligned}\left[\left(\mathbf{u} \cdot \hat{\mathbf{M}}\right) \hat{\mathbf{M}}^{-1}\right] \left[\left(\mathbf{v} \cdot \hat{\mathbf{M}}\right) \hat{\mathbf{M}}^{-1}\right] &= \left[\left(\mathbf{u} \cdot \hat{\mathbf{M}}\right) \hat{\mathbf{M}}^{-1}\right] \left[\hat{\mathbf{M}} \left(\mathbf{v} \cdot \hat{\mathbf{M}}\right)\right] \\ &= \left(\mathbf{u} \cdot \hat{\mathbf{M}}\right) \left(\mathbf{v} \cdot \hat{\mathbf{M}}\right). \end{aligned} \tag{2.7}$$

We could also find $\|\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})\|$ and $\|\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{v})\|$ via a route similar to that used in deriving Eq. (2.7). For example,

$$\begin{aligned}\|\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})\| &= \sqrt{[\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})]^2} \\ &= \sqrt{\left[\left(\mathbf{u} \cdot \hat{\mathbf{M}}\right) \hat{\mathbf{M}}^{-1}\right] \left[\left(\mathbf{u} \cdot \hat{\mathbf{M}}\right) \hat{\mathbf{M}}^{-1}\right]} \\ &= \sqrt{\left(\mathbf{u} \cdot \hat{\mathbf{M}}\right)^2} \\ &= \|\mathbf{u} \cdot \hat{\mathbf{M}}\|.\end{aligned}$$

Now, let's return to the question of $\|\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})\|$ and $\|\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{v})\|$. Because the vector $\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})$ is parallel to the unit bivector $\hat{\mathbf{M}}$, $\left[\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})\right] \hat{\mathbf{M}}$ is just a 90° rotation of $\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})$. Thus, $\|\left[\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})\right] \hat{\mathbf{M}}\| = \|\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})\|$. But $\left[\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})\right] \hat{\mathbf{M}} = \left(\mathbf{u} \cdot \hat{\mathbf{M}}\right) \hat{\mathbf{M}}^{-1} \hat{\mathbf{M}} = \mathbf{u} \cdot \hat{\mathbf{M}}$. After using similar reasoning for $\|\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{v})\|$, we find that

$$\begin{aligned}\|\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{u})\| &= \|\mathbf{u} \cdot \hat{\mathbf{M}}\|, \text{ and} \\ \|\mathsf{P}_{\hat{\mathbf{M}}}(\mathbf{v})\| &= \|\mathbf{v} \cdot \hat{\mathbf{M}}\|.\end{aligned} \tag{2.8}$$

Putting all of these ideas together, plus Eq. (2.5), and recognizing that $\mathbf{u} \cdot \hat{\mathbf{M}}$ and $\mathbf{v} \cdot \hat{\mathbf{M}}$ are vectors (and therefore are of grade 1),

$$\sin\beta = \frac{\hat{\mathbf{M}}^{-1} \langle \left(\mathbf{u} \cdot \hat{\mathbf{M}}\right) \left(\mathbf{v} \cdot \hat{\mathbf{M}}\right) \rangle_2}{\|\mathbf{u} \cdot \hat{\mathbf{M}}\| \|\mathbf{v} \cdot \hat{\mathbf{M}}\|}, \tag{2.9a}$$

$$\cos\beta = \frac{\langle \left(\mathbf{u} \cdot \hat{\mathbf{M}}\right) \left(\mathbf{v} \cdot \hat{\mathbf{M}}\right) \rangle_0}{\|\mathbf{u} \cdot \hat{\mathbf{M}}\| \|\mathbf{v} \cdot \hat{\mathbf{M}}\|}. \tag{2.9b}$$

3 Solutions for $\cos\beta$ and $\sin\beta$

We'll begin by writing $\hat{\mathbf{M}}$ and the two vectors as

$$\begin{aligned}\mathbf{u} &= \hat{\mathbf{a}} u_a + \hat{\mathbf{b}} u_b + \hat{\mathbf{c}} u_c, \\ \mathbf{v} &= \hat{\mathbf{a}} v_a + \hat{\mathbf{b}} v_b + \hat{\mathbf{c}} v_c, \text{ and} \\ \hat{\mathbf{M}} &= \hat{\mathbf{a}}\hat{\mathbf{b}} m_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}} m_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}} m_{ac}.\end{aligned}$$

3.1 Expressions for $\mathbf{u} \cdot \hat{\mathrm{M}}$ and $\mathbf{v} \cdot \hat{\mathrm{M}}$

Vector $\mathbf{u}$ is of grade 1, and bivector $\hat{\mathbf{M}}$ is of grade 2, so from Eq. (2.6),

$$\begin{aligned}\mathbf{u} \cdot \hat{\mathbf{M}} &= \langle \mathbf{u} \hat{\mathbf{M}} \rangle_{2-1} \\ &= \langle \left(\hat{\mathbf{a}} u_a + \hat{\mathbf{b}} u_b + \hat{\mathbf{c}} u_c\right) \left(\hat{\mathbf{a}}\hat{\mathbf{b}} m_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}} m_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}} m_{ac}\right) \rangle_1 \\ &= \hat{\mathbf{a}} \left(-u_b m_{ab} - u_c m_{ac}\right) + \hat{\mathbf{b}} \left(u_a m_{ab} - u_c m_{bc}\right) + \hat{\mathbf{c}} \left(u_a m_{ac} + u_b m_{bc}\right).\end{aligned}$$

Similarly,

$$\mathbf{v}\cdot\hat{\mathbf{M}} = \hat{\mathbf{a}}\left(-v_b m_{ab} - v_c m_{ac}\right) + \hat{\mathbf{b}}\left(v_a m_{ab} - v_c m_{bc}\right) + \hat{\mathbf{c}}\left(v_a m_{ac} + v_b m_{bc}\right).$$

3.2 Expressions for $\|\mathbf{u}\cdot\hat{\mathbf{M}}\|$ and $\|\mathbf{v}\cdot\hat{\mathbf{M}}\|$

Using the expressions that we developed in Section 3.1 , $\|\mathbf{u}\cdot\hat{\mathbf{M}}\|^2$ and $\|\mathbf{v}\cdot\hat{\mathbf{M}}\|$, plus the fact that $m_{ab}^2 + m_{bc}^2 + m_{ac}^2 = 1$ because $\hat{\mathbf{M}}$ is a unit bivector,

$$\begin{aligned}\|\mathbf{u}\cdot\hat{\mathbf{M}}\|^2 = {} & u_a^2\left(1-m_{bc}^2\right) + u_b^2\left(1-m_{ac}^2\right) + u_c^2\left(1-m_{ab}^2\right) \\ & + 2u_a u_b m_{ac} m_{bc} + 2u_b u_c m_{ab} m_{ac} - 2u_a u_c m_{ab} m_{bc}\ ,\end{aligned} \tag{3.1a}$$

and

$$\begin{aligned}\|\mathbf{v}\cdot\hat{\mathbf{M}}\|^2 = {} & v_a^2\left(1-m_{bc}^2\right) + v_b^2\left(1-m_{ac}^2\right) + v_c^2\left(1-m_{ab}^2\right) \\ & + 2v_a v_b m_{ac} m_{bc} + 2v_b v_c m_{ab} m_{ac} - 2v_a v_c m_{ab} m_{bc}\ .\end{aligned} \tag{3.1b}$$

Using the correspondence (Eq. (2.2)) between coefficients in $\hat{\mathbf{M}}$ and $\hat{\mathbf{e}}$, we can also write Eq. (3.1) as

$$\begin{aligned}\|\mathbf{u}\cdot\hat{\mathbf{M}}\|^2 = {} & u_a^2\left(1-e_a^2\right) + u_b^2\left(1-e_b^2\right) + u_c^2\left(1-e_c^2\right) \\ & - 2u_a u_b e_a e_b - 2u_b u_c e_b e_c - 2u_a u_c e_a e_c\ ,\end{aligned} \tag{3.2a}$$

and

$$\begin{aligned}\|\mathbf{v}\cdot\hat{\mathbf{M}}\|^2 = {} & v_a^2\left(1-e_a^2\right) + v_b^2\left(1-e_b^2\right) + v_c^2\left(1-e_c^2\right) \\ & - 2v_a v_b e_a e_b - 2v_b v_c e_b e_c - 2v_a v_c e_a e_c\ .\end{aligned} \tag{3.2b}$$

3.3 Solution for $\sin\beta$

As you might expect, the expansion of Eq. (2.9)a becomes extensive and messy, so we'll do it in several steps.

3.3.1 Expansion of $\langle \left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\rangle_2$

After considerable simplification, using the expressions developed in Section 3.1 for $\mathbf{u}\cdot\hat{\mathbf{M}}$ and $\mathbf{v}\cdot\hat{\mathbf{M}}$,

$$\begin{aligned}\langle \left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\rangle_2 = \hat{\mathbf{a}}\hat{\mathbf{b}}\,&\left[\left(u_a v_b - u_b v_a\right)m_{ab}^2 + \left(u_b v_c - u_c v_b\right)m_{ab}m_{bc}\right.\\ &\left.+\left(u_a v_c - u_c v_a\right)m_{ab}m_{ac}\,\right]\\ +\hat{\mathbf{b}}\hat{\mathbf{c}}\,&\left[\left(u_b v_c - u_c v_b\right)m_{bc}^2 + \left(u_a v_b - u_b v_a\right)m_{ab}m_{bc}\right.\\ &\left.+\left(u_a v_c - u_c v_a\right)m_{bc}m_{ac}\,\right]\\ +\hat{\mathbf{a}}\hat{\mathbf{c}}\,&\left[\left(u_a v_c - u_c v_a\right)m_{ac}^2 + \left(u_a v_b - u_b v_a\right)m_{ab}m_{ac}\right.\\ &\left.+\left(u_b v_c - u_c v_b\right)m_{bc}m_{ac}\,\right].\end{aligned} \tag{3.3}$$

3.3.2 Expansion of $\hat{\mathbf{M}}^{-1}\langle \left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\rangle_2$

Surprisingly, left-multiplying the expression for $\langle \left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\rangle_2$ (Eq. (3.3)) by $\hat{\mathbf{M}}^{-1}$ gives a comparatively simple result. The inverse of a unit bivector is just the negative of that bivector, so $\hat{\mathbf{M}}^{-1} = -\hat{\mathbf{M}} = -\hat{\mathbf{a}}\hat{\mathbf{b}}m_{ab} - \hat{\mathbf{b}}\hat{\mathbf{c}}m_{bc} - \hat{\mathbf{a}}\hat{\mathbf{c}}m_{ac}$. After expanding, carrying out massive cancellations, and using the fact that $m_{ab}^2 + m_{bc}^2 + m_{ac}^2 = 1$,

$$\begin{aligned}\hat{\mathbf{M}}^{-1}\langle \left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\rangle_2 = &\left(u_a v_b - u_b v_a\right)m_{ab} + \left(u_b v_c - u_c v_b\right)m_{bc}\\ &+\left(u_a v_c - u_c v_a\right)m_{ac}.\end{aligned} \tag{3.4}$$

Does our answer make sense?

The expression on the right-hand side of Eq. (3.4) will be the numerator of the final expression for $\sin\beta$. We should stop here to ask ourselves whether that expression makes sense. For example, does it behave as it should, given the physical situation for which it's been derived? One thing we know is that if the vectors $\mathbf{u}$ and $\mathbf{v}$ are reversed, then angle β will remain the same in magnitude, but will change algebraic sign. The same is true for the right-hand side of Eq. (3.4), so in this respect, at least, it does make sense.

We also note that if $\mathbf{u} = \mathbf{v}$, then the right-hand side of Eq. (3.4) is zero, as it should be, because when $\mathbf{u} = \mathbf{v}$, $\beta = 0$.

3.3.3 Final result for $\sin\beta$

Using our result from Eq. (3.4)

$$\sin\beta = \frac{\left(u_a v_b - u_b v_a\right)m_{ab} + \left(u_b v_c - u_c v_b\right)m_{bc} + \left(u_a v_c - u_c v_a\right)m_{ac}}{\|\mathbf{u}\cdot\hat{\mathbf{M}}\|\|\mathbf{v}\cdot\hat{\mathbf{M}}\|}, \tag{3.5}$$

where $\|\mathbf{u}\cdot\hat{\mathbf{M}}\|$ and $\|\mathbf{v}\cdot\hat{\mathbf{M}}\|$ are the square roots of the expressions on the right-hand side of Eqs. (3.1). Using the correspondence (Eq. (2.2)) between coefficients in $\hat{\mathbf{M}}$ and $\hat{\mathbf{e}}$, we can also write Eq. (3.5) as

$$\sin\beta = \frac{(u_a v_b - u_b v_a)\, e_c + (u_b v_c - u_c v_b)\, e_a - (u_a v_c - u_c v_a)\, e_b}{\|\mathbf{u}\cdot\hat{\mathbf{M}}\|\|\mathbf{v}\cdot\hat{\mathbf{M}}\|}. \tag{3.6}$$

3.4 Solution for $\cos\beta$

The work needed to derive the expression for $\cos\beta$ is much less extensive than was needed for $\sin\beta$, so we will omit many of the details.

The expansion of $\left\langle\left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\right\rangle_0$ reduces to

$$\begin{aligned}\left\langle\left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\right\rangle_0 = {} & u_a v_a \left(1 - m_{bc}^2\right) + u_b v_b \left(1 - m_{ac}^2\right) \\ & + u_c v_c \left(1 - m_{ab}^2\right) - (u_a v_c + u_c v_a)\, m_{ab} m_{bc} \\ & + (u_b v_c + u_c v_b)\, m_{ab} m_{ac} + (u_a v_b + u_b v_a)\, m_{bc} m_{ac}.\end{aligned} \tag{3.7}$$

Then, according to Eq. (2.9), $\cos\beta$ is the right-hand side of Eq. (3.7), divided by the product $\|\mathbf{u}\cdot\hat{\mathbf{M}}\|\|\mathbf{v}\cdot\hat{\mathbf{M}}\|$ (from Eqs. (3.1)). Using the correspondence (Eq. (2.2)) between coefficients in $\hat{\mathbf{M}}$ and $\hat{\mathbf{e}}$, we can also write Eq. (3.7) as

$$\begin{aligned}\left\langle\left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\right\rangle_0 = {} & u_a v_a \left(1 - e_a^2\right) + u_b v_b \left(1 - e_b^2\right) \\ & + u_c v_c \left(1 - e_c^2\right) - (u_a v_c + u_c v_a)\, e_a e_c \\ & - (u_b v_c + u_c v_b)\, e_b e_c - (u_a v_b + u_b v_a)\, e_a e_b.\end{aligned} \tag{3.8}$$

Once again, we should ask whether our answer makes sense. For example, does interchanging the vectors $\mathbf{u}$ and $\mathbf{v}$ leave the result unchanged, as it should? Yes.

Does our answer make sense?

In addition, if $\mathbf{u} = \mathbf{v}$, then $\left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\rangle_0$ reduces to the expression that we found for $\|\mathbf{u}\cdot\hat{\mathbf{M}}\|^2$ (Eqs. (3.1)). This result, too, is as it should be.

We could also, more laboriously, verify that $\sin^2\beta + \cos^2\beta = 1$.

4 Testing the Formulas that We've Derived

Fig. 5 shows an interactive GeoGebra construction (Reference [3]) that compares the calculated and actual values of $\sin\beta$ and $\cos\beta$.

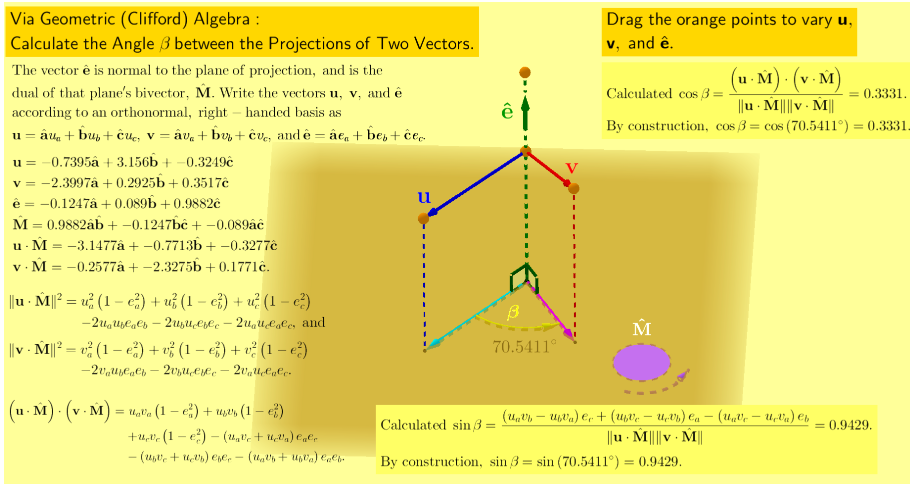

Figure 5: An interactive GeoGebra construction (Reference [3]) that tests the formulas derived herein by comparing the calculated and actual values of $\sin\beta$ and $\cos\beta$.

5 Conclusions, and Summary of the Formulas Derived Herein

The key idea in these derivations was that because the vectors $\left(\mathbf{u} \cdot \hat{\mathbf{M}}\right)\hat{\mathbf{M}}^{-1}$ and $\left(\mathbf{v} \cdot \hat{\mathbf{M}}\right)\hat{\mathbf{M}}^{-1}$ are parallel to $\hat{\mathbf{M}}$, we can calculate the sine and cosine of β from the product $\left(\mathbf{u} \cdot \hat{\mathbf{M}}\right)\left(\mathbf{v} \cdot \hat{\mathbf{M}}\right)$, rather than having to calculate them from the product $\left[\left(\mathbf{u} \cdot \hat{\mathbf{M}}\right)\hat{\mathbf{M}}^{-1}\right]\left[\left(\mathbf{v} \cdot \hat{\mathbf{M}}\right)\hat{\mathbf{M}}^{-1}\right]$.

Writing $\mathbf{u}$, $\mathbf{v}$, and $\hat{\mathbf{e}}$, and $\hat{\mathbf{M}}$, as

- $\mathbf{u} = \hat{\mathbf{a}}u_a + \hat{\mathbf{b}}u_b + \hat{\mathbf{c}}u_c$,
- $\mathbf{v} = \hat{\mathbf{a}}v_a + \hat{\mathbf{b}}v_b + \hat{\mathbf{c}}v_c$,
- $\mathbf{e} = \hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c$, and
- $\hat{\mathbf{M}} = \hat{\mathbf{a}}\hat{\mathbf{b}}m_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}m_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}m_{ac}\ \left(= \hat{\mathbf{a}}\hat{\mathbf{b}}e_c + \hat{\mathbf{b}}\hat{\mathbf{c}}e_a - \hat{\mathbf{a}}\hat{\mathbf{c}}e_b\right)$,

we found that (Eqs. (3.5) and (3.6))

$$\sin\beta = \frac{\left(u_av_b - u_bv_a\right)m_{ab} + \left(u_bv_c - u_cv_b\right)m_{bc} + \left(u_av_c - u_cv_a\right)m_{ac}}{\|\mathbf{u} \cdot \hat{\mathbf{M}}\|\|\mathbf{v} \cdot \hat{\mathbf{M}}\|},$$

$$= \frac{\left(u_av_b - u_bv_a\right)e_c + \left(u_bv_c - u_cv_b\right)e_a - \left(u_av_c - u_cv_a\right)e_b}{\|\mathbf{u} \cdot \hat{\mathbf{M}}\|\|\mathbf{v} \cdot \hat{\mathbf{M}}\|},$$

where $\|\mathbf{u}\cdot\hat{\mathbf{M}}\|$ and $\|\mathbf{v}\cdot\hat{\mathbf{M}}\|$ are the square roots of the expressions on the right-hand side of Eqs. (3.1).

WE also find that (Eqs. (3.7) and (3.8))

$$\cos\beta = \frac{\left\langle\left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\right\rangle_0}{\|\mathbf{u}\cdot\hat{\mathbf{M}}\|\|\mathbf{v}\cdot\hat{\mathbf{M}}\|},$$

where

$$\begin{aligned}\left\langle\left(\mathbf{u}\cdot\hat{\mathbf{M}}\right)\left(\mathbf{v}\cdot\hat{\mathbf{M}}\right)\right\rangle_0 &= u_av_a\left(1-m_{bc}^2\right)+u_bv_b\left(1-m_{ac}^2\right)\\ &\quad+u_cv_c\left(1-m_{ab}^2\right)-\left(u_av_c+u_cv_a\right)m_{ab}m_{bc}\\ &\quad+\left(u_bv_c+u_cv_b\right)m_{ab}m_{ac}+\left(u_av_b+u_bv_a\right)m_{bc}m_{ac},\\[1em] &= u_av_a\left(1-e_a^2\right)+u_bv_b\left(1-e_b^2\right)\\ &\quad+u_cv_c\left(1-e_c^2\right)-\left(u_av_c+u_cv_a\right)e_ae_c\\ &\quad-\left(u_bv_c+u_cv_b\right)e_be_c-\left(u_av_b+u_bv_a\right)e_ae_b.\end{aligned}$$

References

[1] D. Hestenes, 1999, *New Foundations for Classical Mechanics*, (Second Edition), Kluwer Academic Publishers (Dordrecht/Boston/London).

[2] A. Macdonald, *Linear and Geometric Algebra* (First Edition) p. 126, CreateSpace Independent Publishing Platform (Lexington, 2012).

[3] J. A. Smith, 2018, "Angle Between Projections of Vectors via Geom. Algebra" (a GeoGebra construction), https://www.geogebra.org/m/Zpsxygxy.

[4] J. A. Smith, 2017, "Some Solution Strategies for Equations that Arise in Geometric (Clifford) Algebra", http://vixra.org/abs/1610.0054 .

6 Appendix: Calculating the Unit Bivector $\hat{\mathbf{M}}$ of a Plane Whose Dual is the Vector $\hat{\mathbf{e}}$

As may be inferred from a study of References [1] (p. (56, 63) and [2] (pp. 106-108) , the bivector $\hat{\mathbf{M}}$ that we seek is the one whose dual is $\hat{\mathbf{e}}$. That is, $\hat{\mathbf{M}}$ must satisfy the condition

$$\begin{aligned}\hat{\mathbf{e}} &= \hat{\mathbf{M}}I_3^{-1};\\ \therefore\ \hat{\mathbf{M}} &= \hat{\mathbf{e}}I_3.\end{aligned} \tag{6.1}$$

where I_3 is the right-handed pseudoscalar for $\mathbb{G}^3$. That pseudoscalar is the product, written in right-handed order, of our orthonormal reference frame's

Although we won't use that fact here, I_3^{-1} is I_3's negative: $I_3^{-1} = -\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$.

basis vectors: $I_3 = \hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$ (and is also $\hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}\hat{\mathbf{a}}\hat{\mathbf{b}}$). Therefore, writing $\hat{\mathbf{M}}$ as $\hat{\mathbf{M}} = \hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c$,

To make this simplification, we use the following facts:

- The product of two perpendicular vectors (such as $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$) is a bivector;
- Therefore, for any two perpendicular vectors $\mathbf{p}$ and $\mathbf{q}$, $\mathbf{qp} = -\mathbf{qp}$; and
- (Of course) for any unit vector $\hat{\mathbf{p}}$, $\hat{\mathbf{p}}\hat{\mathbf{p}} = 1$.

See also Ref. [4].

$$\begin{aligned}\hat{\mathbf{M}} &= \hat{\mathbf{e}}I_3 \\ &= \left(\hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c\right)\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}} \\ &= \hat{\mathbf{a}}\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}e_a + \hat{\mathbf{b}}\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}e_b + \hat{\mathbf{c}}\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}e_c \\ &= \hat{\mathbf{a}}\hat{\mathbf{b}}e_c + \hat{\mathbf{b}}\hat{\mathbf{c}}e_a - \hat{\mathbf{a}}\hat{\mathbf{c}}e_b. \end{aligned} \tag{6.2}$$

In writing that last result, we've followed [2]'s convention (p. 82) of using $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and $\hat{\mathbf{a}}\hat{\mathbf{c}}$ as our bivector basis. Examining Eq. (6.2) we can see that if we write $\hat{\mathbf{M}}$ in the form $\hat{\mathbf{M}} = \hat{\mathbf{a}}\hat{\mathbf{b}}m_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}m_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}m_{ac}$, then

$$m_{ab} = e_c, \quad m_{bc} = e_a, \quad m_{ac} = -e_b. \tag{6.3}$$