On the Existence of Magnetic Monopoles and the Invariance of Maxwell's equations under Time Reversal

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January 20, 2018

Abstract

Maxwell's equations in Classical Electrodynamics do not contain any magnetic monopole. As a matter of fact there is no experimental evidence of the magnetic monopoles till now. But still the presence of magnetic monopoles explore a wonderful symmetry in Maxwell's Equations and modern theories like String Theory,Grand Unified Theory(GUT) agree with the presence of monopoles. Paul Dirac tried to introduce magnetic monopoles through the idea of 'Dirac String'.In this paper we will find that magnetic monopoles can be introduced without affecting the classical framework just by defining two auxiliary fields which behave just like electric($\mathbf{E}$)and magnetic($\mathbf{B}$) fields. Besides these auxiliary fields can also make Maxwell's Equations symmetric under time reversal.

0.1 Introduction

Symmetry is a profound concept in fundamental laws of nature. Maxwell's equations are the following(in C.G.S),

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \tag{3}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}_e \tag{4}$$

Clearly there is no symmetry within these four equations. But if we can include the magnetic monopole term, then

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e \tag{5}$$

$$\nabla \cdot \mathbf{B} = 4\pi\rho_m \tag{6}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\frac{4\pi}{c}\mathbf{J}_m \tag{7}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}_e \tag{8}$$

Now these four equations become perfectly symmetric under the duality transformation,

E→**B** ; **B**→-**E**
$\rho_e \rightarrow \rho_m$; $\rho_m \rightarrow$ - ρ_e
Je→ **Jm** ; **Jm** → **-Je**

But there is a major flaw in this formulation. Although at first the scalar(V) and vector(**A**) potential may seem to be just mathematical tools, in reality they do appear in physical world. From Aharonov-Bohm effect we get to know that the vector potential **A** has in fact a physical significance. But the origin of **A** is,

$$\nabla \cdot \mathbf{B} = 0 \tag{9}$$

As a result we could express **B** as a curl of a vector ,

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{10}$$

But if we include magnetic monopole terms in Maxwell's equations then

$$\nabla \cdot \mathbf{B} = 4\pi\rho_m \tag{11}$$

Now we cannot introduce **A**. But Aharonov-Bohm effect confirms the existence of **A**. So this becomes a contradiction.

0.2 The Idea of Dirac String

In 1931 P.A.M Dirac showed that it is possible to prove the presence of both magnetic monopoles and vector potentials **A**. He also showed that the existence of magnetic monopole guarantees the quantisation of electric charge. In nature electric charge is always quantised. So it is a prime evidence of the existence of magnetic monopoles. Dirac considered a spherical arbitrary surface and assumed **A** to be non-singular everywhere except at the origin where the magnetic charge is situated. From that he showed that if we consider concentric spheres around the original sphere then an infinite string passing through the origin can exist, on which **A** is not defined. It is known as the Dirac String. So this idea beautifully combines the existence of magnetic monopoles and vector potential in Maxwell's equations.

0.3 An Alternative approach - Idea of Auxiliary Fields

From Maxwell's original equations (1),(2),(3),(4) we can express **E** and **B** as following,

$$\mathbf{E} = -\nabla \mathbf{V} \;\; ; \;\; \mathbf{B} = \nabla \times \mathbf{A} \tag{12}$$

The four equations of Maxwell can be expressed by the two potentials (V and **A**) by the following two equations,

$$-\nabla^2 \mathbf{V} - \frac{1}{c}\frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = 4\pi\rho_e \tag{13}$$

$$(\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}) - \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \mathbf{V}}{\partial t}) = -\frac{4\pi}{c}\mathbf{J}_e \tag{14}$$

But after including magnetic monopole terms in Maxwell's equations we cannot define **A** because,

$$\nabla \cdot \mathbf{B} \neq 0 \tag{15}$$

But we can write **B** as following,

$$\mathbf{B} = \nabla \times \mathbf{A} + \mathbf{S} \tag{16}$$

where **S** is another vector field so that,

$$\nabla \cdot \mathbf{S} = 4\pi\rho_m \tag{17}$$

Now let us put (16) into (7),

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial(\nabla \times \mathbf{A} + \mathbf{S})}{\partial t} = -\frac{4\pi}{c}\mathbf{J}_m \tag{18}$$

$$\Rightarrow \nabla \times (\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) = -(\frac{4\pi}{c}\mathbf{J}_m + \frac{1}{c}\frac{\partial \mathbf{S}}{\partial t}) \tag{19}$$

Now doing the same trick again we get,

$$(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) = -\nabla \mathbf{V} + \mathbf{S^*} \tag{20}$$

where **S*** is yet another field such that,

$$\nabla \times \mathbf{S^*} = -(\frac{4\pi}{c}\mathbf{J}_m + \frac{1}{c}\frac{\partial \mathbf{S}}{\partial t}) \tag{21}$$

Now the selection of **S** and **S*** is arbitrary but by closely inspecting (17) and (21) we realise that **S** and **S*** satisfy Maxwell's second and third equations respectively and **S** works like **B** whereas **S*** works like **E** though,

$$\mathbf{S} \neq \mathbf{B} \;\; ; \;\; \mathbf{S^*} \neq \mathbf{E} \tag{22}$$

If we assume that **S** and **S*** also satisfy the other two Maxwell's equations then the new form of (13) and (14) becomes

$$-\nabla^2 \mathbf{V} - \frac{1}{c}\frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = 0 \tag{23}$$

$$(\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}) - \nabla(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\mathbf{V}}{\partial t}) = 0 \tag{24}$$

which is exactly the form of (13) and (14) in absence of any electric and magnetic charge.

So from now on we will use **B*** and **E*** in place of **S** and **S*** respectively. These **B*** and **E*** can be called auxiliary fields which also satisfy the Maxwell's equations.

0.4 Solution to time reversal asymmetry of Maxwell's equations

The original Maxwell's equations(the ones without the magnetic monopoles) are symmetric under time reversal. This means if we turn back time then the laws would be exactly the same. Mathematically we can do this by changing t→-t. The other elements change under time reversal as following

E→**E** ; **B**→ -**B**

$\rho_e \rightarrow \rho_e$; **Je**→ -**Je**

Maxwell's equations remain invariant under these changes. But if we try to apply time reversal on the Maxwell's equations containing magnetic monopoles then,

$$\nabla\cdot\mathbf{E}_+ = 4\pi\rho_e + \tag{25}$$

$$\nabla\cdot\mathbf{B}_- = 4\pi\rho_m ? \tag{26}$$

$$\nabla\times\mathbf{E}_+ + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}_+ = -\frac{4\pi}{c}\mathbf{J}_m ? \tag{27}$$

$$\nabla\times\mathbf{B}_- - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}_- = \frac{4\pi}{c}\mathbf{J}_e - \tag{28}$$

Clearly the modified form of Maxwell's equations are time reversal asymmetric unless we consider,

$\rho_m \rightarrow$ -ρ_m ; **Jm**→ **Jm**

Now this is quite counter-intuitive, as under time reversal,

$\rho_e \rightarrow \rho_e$; **Jm**→ -**Jm**

If the magnetic charge density and magnetic current density depend on some other fields that reverse sign under time reversal just like **E** and **B** , the Maxwell's equations will again be symmetric. It may sound nonphysical but we cannot strikeout the possibility. Now from earlier discussion we've got auxiliary fields that can serve this purpose. Now,

$$\nabla\cdot\mathbf{B^*} = 4\pi\rho_m \tag{29}$$

$$\nabla\times\mathbf{E^*} + \frac{1}{c}\frac{\partial\mathbf{B^*}}{\partial t} = -\frac{4\pi}{c}\mathbf{J}_m \tag{30}$$

Let, under time reversal

$\rho_m \rightarrow \rho_m*$; $\mathbf{Jm}\rightarrow \mathbf{Jm}^*$

So under time reversal (29) becomes,

$$\nabla \cdot \mathbf{(\text{-}B^*)} = 4\pi\rho_m* \tag{31}$$

$$\Rightarrow -\nabla \cdot \mathbf{B^*} = 4\pi\rho_m* \tag{32}$$

$$\Rightarrow -4\pi\rho_m = 4\pi\rho_m* \tag{33}$$

$$\Rightarrow \rho_m* = -\rho_m \tag{34}$$

In a similar way from (30) we get

$$\mathbf{J^*}_m = \mathbf{J}_m \tag{35}$$

(34) and (35) clearly satisfies the condition for the invariance of time reversal of Maxwell's modified equations.

0.5 Conclusion

So the consideration of the auxiliary fields is a good alternate to Dirac String as it not only solves the problem with the existence of magnetic vector potential **A** but also makes sure that Maxwell's equations remain time invariant in all cases. Although the idea of auxiliary fields is totally mathematical, its existence cannot be denied just because it is not proven experimentally. Besides it shows how simple yet beautiful the existence of magnetic monopoles can be in Classical framework.

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