SMARANDACHE BCI-ALGEBRAS

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Abstract. The notion of Smarandache (positive implicative, commutative, implicative) BCI-algebras, Smarandache subalgebras and Smarandache ideals is introduced, examples are given, and related properties are investigated.

1. INTRODUCTION

Generally, in any human field, a Smarandache Structure on a set \( A \) means a weak structure \( W \) on \( A \) such that there exists a proper subset \( B \) of \( A \) which is embedded with a strong structure \( S \). In [6], W. B. Vasanth Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids, semi-normal subgroupoids, Smarandache Bol groupoids and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and it was studied by R. Padilla [5]. It will be very interesting to study the Smarandache structure in \( BCK/BCI \)-algebras. Thus, in this paper, we discuss the Smarandache structure in \( BCI \)-algebras. We introduce the notion of Smarandache (positive implicative, commutative, implicative) \( BCI \)-algebras, Smarandache subalgebras and Smarandache ideals, and investigate some related properties.

2. PRELIMINARIES

An algebra \( (X, *, 0) \) of type \((2,0)\) is called a \textit{BCI-algebra} if it satisfies the following conditions:

(I) \( \forall x, y, z \in X \) \( ((x * y) * (z * y)) * (z * y) = 0 \),

(II) \( \forall x, y \in X \) \( ((x * (x * y)) * y = 0 \),

(III) \( \forall x \in X \) \( (x * x = 0) \),

(IV) \( \forall x, y \in X \) \( (x * y = 0, y * x = 0 \Rightarrow z = y) \).

If a \( BCI \)-algebra \( X \) satisfies the following identity:

(V) \( \forall x \in X \) \( (0 * x = 0) \),

then \( X \) is called a \textit{BCK-algebra}. We can define a partial order ‘\( \leq \)’ on \( X \) by \( x \leq y \) if and only if \( x * y = 0 \). Every \( BCI \)-algebra \( X \) has the following properties:

(a1) \( \forall x \in X \) \( (x * 0 = x) \).

(a2) \( \forall x, y \in X \) \( ((x * y) * z = (x * z) * y) \).

(a3) \( \forall x, y, z \in X \) \( (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x) \).

A \( BCI \)-algebra \( X \) is called a \textit{medial BCI-algebra} if it satisfies:

\( \forall x, y, z, u \in X \) \( ((x * y) * (z * u) = (x * z) * (y * u)) \).

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For a $BCI$-algebra $X$, the set $X_+ := \{ x \in X \mid 0 \leq x \}$ is called the $BCK$-part of $X$. If $X_+ = \{0\}$, then $X$ is called a $p$-semisimple $BCI$-algebra. Note that $X$ is a medial $BCI$-algebra if and only if $X$ is a $p$-semisimple $BCI$-algebra. A $BCI$-algebra $X$ is said to be associative if it satisfies:

$$((\forall x, y, z \in X) ((x \ast y) \ast z = x \ast (y \ast z))).$$

Every associative $BCI$-algebra is a $p$-semisimple $BCI$-algebra. A nonempty subset $I$ of a $BCI$-algebra $X$ is called an ideal of $X$ if it satisfies the following conditions:

(i) $0 \in I$,
(ii) $(\forall x \in X) (\forall y \in I) (x \ast y \in I \Rightarrow x \in I)$.

3. SMARANDACHE $BCI$-ALGEBRAS

A Smarandache $BCI$-algebra is defined to be a $BCI$-algebra $X$ in which there exists a proper subset $Q$ of $X$ such that

- $0 \in Q$ and $|Q| \geq 2$,
- $Q$ is a $BCK$-algebra under the operation of $X$.

By a Smarandache positive implicative (resp. commutative and implicative) $BCI$-algebra, we mean a $BCI$-algebra $X$ which has a proper subset $Q$ of $X$ such that

- $0 \in Q$ and $|Q| \geq 2$,
- $Q$ is a positive implicative (resp. commutative and implicative) $BCK$-algebra under the operation of $X$.

Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</table>

Table 1

Then $(X; \ast, 0)$ is a Smarandache $BCI$-algebra. Let $X_1 = \{0, a, b\}$ and $X_2 = \{0, a, b, c\}$ be sets with the following Cayley tables:

<table>
<thead>
<tr>
<th>$\ast_1$</th>
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<th>b</th>
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<table>
<thead>
<tr>
<th>$\ast_2$</th>
<th>0</th>
<th>a</th>
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<tr>
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<td>c</td>
<td>b</td>
<td>a</td>
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</tbody>
</table>

Table 2 Table 3

Then $(X_1; \ast_1, 0)$ and $(X_2; \ast_2, 0)$ are not Smarandache $BCI$-algebras. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:
Then \((X; *, O)\) is a Smarandache implicative BCI-algebra, because \((Q := \{0, 1, 2\}; *, O)\) is an implicative BCK-algebra. Let \(X = \{0, 1, 2, 3, 4, 5\}\) be a set with the following Cayley table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
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</table>

Table 4

Then \((X; *, O)\) is a Smarandache positive implicative BCI-algebra, because \((Q := \{0, 1, 2, 3\}; *, O)\) is a positive implicative BCK-algebra. Let \(X = \{0, 1, 2, 3, 4, 5\}\) be a set with the following Cayley table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
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</table>

Table 5

Then \((X; *, O)\) is a Smarandache commutative BCI-algebra, because \((Q := \{0, 1, 2, 3\}; *, O)\) is a commutative BCK-algebra. Using the fact that a BCK-algebra is implicative if and only if it is both commutative and positive implicative, we know that a BCI-algebra \(X\) is a Smarandache implicative BCI-algebra if and only if \(X\) is both a Smarandache commutative BCI-algebra and a Smarandache positive implicative BCI-algebra. For a BCI-algebra \(X\), if \(|X_+| = 1\), then there is no non-trivial proper subset \(Q\) of \(X\) which is a BCK-algebra under the operation of \(X\). Hence any BCI-algebra \(X\) with \(|X_+| = 1\) cannot be a Smarandache BCI-algebra. Using this result, we know that every p-semisimple or medial BCI-algebra cannot be a Smarandache BCI-algebra, and every associative BCI-algebra cannot be a Smarandache BCI-algebra. We also note that if a BCI-algebra \(X\) satisfies one of the following assertions:

(i) \(\forall x, y \in X\) \((x \ast (x \ast y) = y)\),
(ii) \(\forall x, y \in X\) \((x \ast y = 0 \ast (y \ast x))\),
(iii) \(\forall x, y, z \in X\) \((x \ast (y \ast z) = z \ast (y \ast z))\),
(iv) \(\forall x \in X\) \((0 \ast x = 0 \Rightarrow x = 0)\),
(v) \((\forall x \in X) \ (0 * (0 * x) = x)\),
(vi) \((\forall x, y \in X) \ (x * (0 * y) = y * (0 * x))\),
(vii) \((\forall x, y, z \in X) \ ((x * y) * z = 0 * ((y * (0 * x)) * x))\),
(viii) \((\forall x, y, z \in X) \ ((x * y) * (z * x) = z * y)\),
(ix) \((\forall x, y, z, u \in X) \ ((x * u) * (z * y) = (y * u) * (z * x))\),
(x) \((\forall x, y, z \in X) \ ((0 * y) * (0 * x) = x * y)\),
(xi) \((\forall x, y \in X) \ (0 * (0 * (x * y)) = x * y)\),
(xii) \((\forall x, y \in X) \ (z * (z * (x * y)) = x * y)\),
(xiii) \((\forall x, y \in X) \ (x * y = 0 \Rightarrow x = y)\),
(xiv) \((\forall x, y, z \in X) \ (x * y = x * z \Rightarrow y = z)\),
(xv) \((\forall x, y, z \in X) \ (z * x = z * y \Rightarrow x = y)\),
(xvi) \((\forall x, y, z \in X) \ ((x * y) * (z * z) = 0 * (y * z))\),
(xvii) \((\forall x, y, z \in X) \ (x * y = 0 \Rightarrow (z * x) * (z * y) = 0)\),
(xviii) \((\forall x, y, z \in X) \ ((z * z) * (z * y) = 0 \Rightarrow z * y = 0)\),
(xix) \((\forall x, y, z \in X) \ (x * y = 0 \Rightarrow (y * z) * (z * z) = 0)\),
(xx) \((\forall x, y, z \in X) \ ((z * z) * (y * z) = 0 \Rightarrow x * y = 0)\),
(xxi) \((\forall x, y, z, w \in X) \ ((x * y) * (z * w) = (w * z) * (y * x))\),
(xxii) \((\forall x, y, z \in X) \ ((x * y) * z = (0 * z) * (y * z))\),
(xxiii) \((\forall x, y, z \in X) \ (x * (y * z) = (z * y) * (0 * z))\),

then \(X\) cannot be a Smarandache BCI-algebra. Let \(X\) be a BCI-algebra that satisfies the identity \(0 * x = x\) for all \(x \in X\). If \(X\) satisfies one of the following assertions:

(i) \((\forall x, y, z \in X) \ (x * (y * z) = (z * y) * x)\),
(ii) \((\forall x, y \in X) \ (x * y = x)\),
(iii) \((\forall x, y, z \in X) \ ((x * y) * z = (z * y) * x)\),
(iv) \((\forall x, y, z, u \in X) \ ((x * y) * (z * u) = (x * z) * (y * u))\),
(v) \((\forall x, y, z \in X) \ (x * (x * y) = y)\),

then \(X\) cannot be a Smarandache BCI-algebra. Also, an algebra \((X; *, 0)\) of type (2, 0)
that satisfies the following conditions:

(i) \((\forall x, y, z \in X) \ ((x * 0) * (y * z) = z * (y * z))\),
(ii) \((\forall x, y \in X) \ (x * (y * y) = x)\)
cannot be a Smarandache BCI-algebra, and an algebra \((X; *, 0)\) of type (2, 0) that satisfies
the following conditions:

(i) \((\forall x \in X) \ (x * x = 0)\),
(ii) \((\forall x, y \in X) \ (x * (0 * y) = x)\),
(iii) \((\forall x, y, z, w \in X) \ ((x * y) * (w * z) = (x * w) * (y * z))\),
(v) \((\forall x, y, z \in X) \ ((x * y) * (x * z) = z * y)\)
cannot be a Smarandache BCI-algebra.

A Cayley table of a set \(X\) is said to be of type \(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\) if the \((1, 1)\), \((1, 2)\) and \((2, 2)\)-
entry is 0, and the \((2, 1)\)-entry is 1. For example, tables 4, 5, and 6 are of type \(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\).

We know that every BCI-algebra with a Cayley table of type \(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\) is a Smarandache
BCI-algebra, and every BCI-algebra \(X\) with \(|X| \geq 2\) is a Smarandache BCI-algebra. Let \(X\) be a nontrivial BCK-algebra and let \(w\) be an ideal element which is not contained in \(X\). If we define \(x * w = w * x = w\) for any \(x \in X\) and \(w * w = 0\), then \(X \cup \{w\}\) is a
BCI-algebra, and so it is a Smarandache BCI-algebra. It is well known that any group \(G\)
in which the square of every element is the identity \(e\) is a BCI-algebra and such a group \(G\)
belongs to the variety of quasi-commutative $BCI$-algebras of type $(1,0;0,0)$. The group $G$ is used to make a new $BCI$-algebra together with a $BCK$-algebra. Let $X$ be a nontrivial $BCK$-algebra and let $Y = (G \setminus \{e\}) \cup X$. We define the operation $*$ on $Y$ as follows:

(i) For $x, y \in G \setminus \{e\}$, we put
$$x * y := \begin{cases} xy \in G \setminus \{e\} & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

(ii) For $x, y \in X$, we put
$$x * y = x * y \in X.$$

(iii) For $x \in G \setminus \{e\}$ and $y \in X$, we put
$$x * y = y * x = x.$$

Then $Y$ is a $BCI$-algebra, and so it is a Smarandache $BCI$-algebra.

For a nontrivial $BCK$-algebra $(X_1; *, 0)$ and a $BCI$-algebra $(X_2; *, 0)$, let $X = X_1 \cup X_2$ and define a binary operation $*$ on $X$ as follows:

$$x * y := \begin{cases} x_1 * y & \text{if } x, y \in X_1, \\ x_2 * y & \text{if } x, y \in X_2, \\ 0 & \text{if } x \in X_1, y \in X_2, y \neq 0, \\ x & \text{if } x \in X_1, y \in X_2, y = 0, \end{cases}$$

Then $(X; *, 0)$ is a $BCI$-algebra, and thus it is a Smarandache $BCI$-algebra.

Let $(X; *, 0)$ be a Smarandache $BCI$-algebra and let $H$ be a subset of $X$ such that $0 \in H$ and $|H| \geq 2$. Then $H$ is called a Smarandache subalgebra of $X$. For example, consider a Smarandache $BCI$-algebra $X = \{0,1,2,3,4,5\}$ with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
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Table 7

Then $H_1 = \{0,1,2,3\}$ is a Smarandache subalgebra of $X$. Any subalgebra of a Smarandache $BCI$-algebra $X$ need not in general be a Smarandache subalgebra of $X$. For example, in the Smarandache $BCI$-algebra $X = \{0,1,2,3,4,5\}$ with the Table 7, the set $H_2 = \{0,2,3\}$ is a subalgebra of $X$ which is not a Smarandache subalgebra of $X$. For a Smarandache $BCI$-algebra $X$, let $H$ be a subalgebra of $X$. If $H$ have a Cayley table of type

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then $H$ is a Smarandache subalgebra of $X$. If a $BCI$-algebra $(X; *, 0)$ contains a Smarandache subalgebra, then $X$ is a Smarandache $BCI$-algebra. In fact, let $H$ be a Smarandache subalgebra of $X$. Then there is a proper subset $Q$ of $H$ such that $0 \in Q$, $|Q| \geq 2$ and $(Q; *, 0)$ is a $BCK$-algebra. Since $H \subseteq X$, it follows that $X$ is a Smarandache $BCI$-algebra. Let $X$ be a Smarandache $BCI$-algebra. A nonempty subset $I$ of $X$ is called a Smarandache ideal of $X$ related to $Q$ if it satisfies:

(i) $0 \in I$,

(ii) $(\forall x \in Q) (\forall y \in I) (x * y \in I \Rightarrow x \in I),$
where $Q$ is a $BCK$-algebra contained in $X$. If $I$ is a Smarandache ideal of $X$ related to every $BCK$-algebra contained in $X$, we simply say that $I$ is a Smarandache ideal of $X$. Since $X_+$ is a maximal $BCK$-algebra contained in $X$, every subset $I$ of a Smarandache $BCI$-algebra $X$ containing $X_+$ is a Smarandache ideal of $X$. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a Smarandache $BCI$-algebra with the table 5. Then subsets $I = \{0, 1, 2\}$ and $J = \{0, 1, 3\}$ of $X$ are Smarandache ideals of $X$ related to a $BCK$-algebra $Q = \{0, 1, 2, 3\}$ with respect to the operation $*$ on $X$. Let $Q_1$ and $Q_2$ be $BCK$-algebras contained in a Smarandache $BCI$-algebra $X$ and $Q_1 \subseteq Q_2$. Then every Smarandache ideal of $X$ related to $Q_2$ is a Smarandache ideal of $X$ related to $Q_1$, but the converse is not true. For example, consider $BCK$-algebras $Q_1 = \{0, 1, 2\}$ and $Q_2 = \{0, 1, 2, 3\}$ in a Smarandache $BCI$-algebra $X = \{0, 1, 2, 3, 4, 5\}$ with the table 5. Then a subset $I = \{0, 2, 3\}$ is a Smarandache ideal of $X$ related to $Q_1$, but not a Smarandache ideal of $X$ related to $Q_2$ since $1 \ast 2 = 0 \in I$ and $2 \in I$ but $1 \notin I$. Thus we know that there exists a $BCK$-algebra $Q$ contained in a Smarandache $BCI$-algebra $X$ such that a Smarandache ideal of $X$ related to $Q$ is not an ideal of $X$.

References


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