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SMARANDACHE *BCI*-ALGEBRAS

YOUNG BAE JUN

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ABSTRACT. The notion of Smarandache (positive implicative, commutative, implicative) BCI-algebras, Smarandache subalgebras and Smarandache ideals is introduced, examples are given, and related properties are investigated.

1. INTRODUCTION

Generally, in any human field, a *Smarandache Structure* on a set A means a weak structure **W** on A such that there exists a proper subset B of A which is embedded with a strong structure S. In [6], W. B. Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids, semi-normal subgroupoids, Smarandache Bol groupoids and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and it was studied by R. Padilla [5]. It will be very interesting to study the Smarandache structure in BCK/BCI-algebras. Thus, in this paper, we discuss the Smarandache structure in BCI-algebras. We introduce the notion of Smarandache (positive implicative, commutative, implicative) BCI-algebras, Smarandache subalgebras and Smarandache ideals, and investigate some related properties.

2. PRELIMINARIES

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x,y,z \in X)\ (((x*y)*(x*z))*(z*y)=0)$,
(II) $(\forall x,y \in X)\ ((x*(x*y))*y=0)$,
(III) $(\forall x \in X)\ (x*x=0)$,
(IV) $(\forall x,y \in X)\ (x*y=0, y*x=0 \Rightarrow x=y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0*x=0)$,

then X is called a *BCK-algebra*. We can define a partial order '$\leq$' on X by $x \leq y$ if and only if $x*y=0$. Every BCI-algebra X has the following properties:

(a1) $(\forall x \in X)\ (x*0=x)$.
(a2) $(\forall x,y,z \in X)\ ((x*y)*z=(x*z)*y)$.
(a3) $(\forall x,y,z \in X)\ (x \leq y \Rightarrow x*z \leq y*z,\ z*y \leq z*x)$.

A BCI-algebra X is called a *medial BCI-algebra* if it satisfies:

$$(\forall x,y,z,u \in X)\ ((x*y)*(z*u)=(x*z)*(y*u)).$$

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For a BCI-algebra X, the set $X_+ := \{x \in X \mid 0 \leq x\}$ is called the *BCK-part* of X. If $X_+ = \{0\}$, then X is called a *p-semisimple BCI-algebra*. Note that X is a medial BCI-algebra if and only if X is a p-semisimple BCI-algebra. A BCI-algebra X is said to be *associative* if it satisfies:

$$(\forall x, y, z \in X)\,((x * y) * z = x * (y * z)).$$

Every associative BCI-algebra is a p-semisimple BCI-algebra. A nonempty subset I of a BCI-algebra X is called an *ideal* of X if it satisfies the following conditions:

(i) $0 \in I$,
(ii) $(\forall x \in X)\ (\forall y \in I)\ (x * y \in I \Rightarrow x \in I)$.

3. Smarandache BCI-algebras

A *Smarandache BCI-algebra* is defined to be a BCI-algebra X in which there exists a proper subset Q of X such that

- $0 \in Q$ and $|Q| \geq 2$,
- Q is a BCK-algebra under the operation of X.

By a *Smarandache positive implicative* (resp. *commutative* and *implicative*) *BCI-algebra,* we mean a BCI-algebra X which has a proper subset Q of X such that

- $0 \in Q$ and $|Q| \geq 2$,
- Q is a positive implicative (resp. commutative and implicative) BCK-algebra under the operation of X.

Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	3	2	3	3
1	1	0	3	2	3	3
2	2	2	0	3	0	0
3	3	3	2	0	2	2
4	4	2	1	3	0	1
5	5	2	1	3	1	0

Table 1

Then $(X; *, 0)$ is a Smarandache BCI-algebra. Let $X_1 = \{0, a, b\}$ and $X_2 = \{0, a, b, c\}$ be sets with the following Cayley tables:

$*_1$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Table 2

$*_2$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Table 3

Then $(X_1; *_1, 0)$ and $(X_2; *_2, 0)$ are not Smarandache BCI-algebras. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	1	3	3	3
2	2	2	0	3	3	3
3	3	3	3	0	0	0
4	4	3	4	1	0	0
5	5	3	5	1	1	0

Table 4

Then $(X; *, 0)$ is a Smarandache implicative BCI-algebra, because $(Q := \{0, 1, 2\}; *, 0)$ is an implicative BCK-algebra. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	0	1	4	4
2	2	2	0	2	4	4
3	3	3	3	0	4	4
4	4	4	4	4	0	0
5	5	4	4	5	1	0

Table 5

Then $(X; *, 0)$ is a Smarandache positive implicative BCI-algebra, because $(Q := \{0, 1, 2, 3\}; *, 0)$ is a positive implicative BCK-algebra. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	0	1	4	4
2	2	1	0	2	4	4
3	3	3	3	0	4	4
4	4	4	4	4	0	0
5	5	4	4	5	1	0

Table 6

Then $(X; *, 0)$ is a Smarandache commutative BCI-algebra, because $(Q := \{0, 1, 2, 3\}; *, 0)$ is a commutative BCK-algebra. Using the fact that a BCK-algebra is implicative if and only if it is both commutative and positive implicative, we know that a BCI-algebra X is a Smarandache implicative BCI-algebra if and only if X is both a Smarandache commutative BCI-algebra and a Smarandache positive implicative BCI-algebra. For a BCI-algebra X, if $|X_+| = 1$, then there is no non-trivial proper subset Q of X which is a BCK-algebra under the operation of X. Hence any BCI-algebra X with $|X_+| = 1$ cannot be a Smarandache BCI-algebra. Using this result, we know that every p-semisimple or medial BCI-algebra cannot be a Smarandache BCI-algebra, and every associative BCI-algebra cannot be a Smarandache BCI-algebra. We also note that if a BCI-algebra X satisfies one of the following assertions:

(i) $(\forall x, y \in X)\ (x * (x * y) = y)$,
(ii) $(\forall x, y \in X)\ (x * y = 0 * (y * x))$,
(iii) $(\forall x, y, z \in X)\ (x * (y * z) = z * (y * x))$,
(iv) $(\forall x \in X)\ (0 * x = 0 \Rightarrow x = 0)$,

(v) $(\forall x \in X)\ (0 * (0 * x) = x)$,
(vi) $(\forall x, y \in X)\ (x * (0 * y) = y * (0 * x))$,
(vii) $(\forall x, y, z \in X)\ ((x * y) * z = 0 * ((y * (0 * z)) * x))$,
(viii) $(\forall x, y, z \in X)\ ((z * y) * (z * x) = x * y)$,
(ix) $(\forall x, yz, u \in X)\ ((x * u) * (z * y) = (y * u) * (z * x))$,
(x) $(\forall x, y \in X)\ ((0 * y) * (0 * x) = x * y)$,
(xi) $(\forall x, y \in X)\ (0 * (0 * (x * y)) = x * y)$,
(xii) $(\forall x, y \in X)\ (z * (z * (x * y)) = x * y)$,
(xiii) $(\forall x, y \in X)\ (x * y = 0 \Rightarrow x = y)$,
(xiv) $(\forall x, y, z \in X)\ (x * y = x * z \Rightarrow y = z)$,
(xv) $(\forall x, y, z \in X)\ (z * x = z * y \Rightarrow x = y)$,
(xvi) $(\forall x, y, z \in X)\ ((x * y) * (x * z) = 0 * (y * z))$,
(xvii) $(\forall x, y, z \in X)\ (x * y = 0 \Rightarrow (z * x) * (z * y) = 0)$,
(xviii) $(\forall x, y, z \in X)\ ((z * x) * (z * y) = 0 \Rightarrow x * y = 0)$,
(xix) $(\forall x, y, z \in X)\ (x * y = 0 \Rightarrow (y * z) * (x * z) = 0)$,
(xx) $(\forall x, y, z \in X)\ ((x * z) * (y * z) = 0 \Rightarrow x * y = 0)$,
(xxi) $(\forall x, y, z, w \in X)\ ((x * y) * (z * w) = (w * z) * (y * x))$,
(xxii) $(\forall x, y, z \in X)\ ((x * y) * z = (0 * z) * (y * x))$,
(xxiii) $(\forall x, y, z \in X)\ (x * (y * z) = (z * y) * (0 * x))$,

then X cannot be a Smarandache BCI-algebra. Let X be a BCI-algebra that satisfies the identity $0 * x = x$ for all $x \in X$. If X satisfies one of the following assertions:

(i) $(\forall x, y, z \in X)\ (x * (y * z) = (x * y) * z)$,
(ii) $(\forall x, y \in X)\ ((x * y) * y = x)$,
(iii) $(\forall x, y, z \in X)\ ((x * y) * z = (z * y) * x)$,
(iv) $(\forall x, y, z, u \in X)\ ((x * y) * (z * u) = (x * z) * (y * u))$,
(v) $(\forall x, y, z \in X)\ (x * (x * y) = y)$,

then X cannot be a Smarandache BCI-algebra. Also, an algebra $(X; *, 0)$ of type $(2, 0)$ that satisfies the following conditions:

(i) $(\forall x, y, z \in X)\ ((x * 0) * (y * z) = z * (y * x))$,
(ii) $(\forall x, y \in X)\ (x * (y * y) = x)$

cannot be a Smarandache BCI-algebra, and an algebra $(X; *, 0)$ of type $(2, 0)$ that satisfies the following conditions:

(i) $(\forall x \in X)\ (x * x = 0)$,
(ii) $(\forall x, y \in X)\ (x * y = 0 = y * x \Rightarrow x = y)$,
(iii) $(\forall x, y, z, w \in X)\ ((x * y) * (w * z) = (x * w) * (y * z))$,
(v) $(\forall x, y, z \in X)\ ((x * y) * (x * z) = z * y)$

cannot be a Smarandache BCI-algebra.

A Cayley table of a set X is said to be of *type* $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ if the $(1, 1)$, $(1, 2)$ and $(2, 2)$-entry is 0, and the $(2, 1)$-entry is 1. For example, tables 4, 5, and 6 are of type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We know that every BCI-algebra with a Cayley table of type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is a Smarandache BCI-algebra, and every BCI-algebra X with $|X_+| \geq 2$ is a Smarandache BCI-algebra. Let X be a nontrivial BCK-algebra and let w be an ideal element which is not contained in X. If we define $x * w = w * x = w$ for any $x \in X$ and $w * w = 0$, then $X \cup \{w\}$ is a BCI-algebra, and so it is a Smarandache BCI-algebra. It is well known that any group G in which the square of every element is the identity e is a BCI-algebra and such a group G

belongs to the variety of quasi-commutative BCI-algebras of type $(1,0;0,0)$. The group G is used to make a new BCI-algebra together with a BCK-algebra. Let X be a nontrivial BCK-algebra and let $Y = (G \setminus \{e\}) \cup X$. We define the operation $*$ on Y as follows:

(i) For $x, y \in G \setminus \{e\}$, we put

$$x * y := \begin{cases} xy \text{ in } G \setminus \{e\} & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

(ii) For $x, y \in X$, we put

$$x * y = x * y \text{ in } X.$$

(iii) For $x \in G \setminus \{e\}$ and $y \in X$, we put

$$x * y = y * x = x.$$

Then Y is a BCI-algebra, and so it is a Smarandache BCI-algebra.

For a nontrivial BCK-algebra $(X_1; *_1, 0)$ and a BCI-algebra $(X_2; *_2, 0)$, let $X = X_1 \cup X_2$ and define a binary operation $*$ on X as follows:

$$x * y := \begin{cases} x *_1 y & \text{if } x, y \in X_1, \\ x *_2 y & \text{if } x, y \in X_2, \\ x & \text{if } x \in X_2, y \in X_1, \\ 0 *_2 y & \text{if } x \in X_1, y \in X_2, y \neq 0, \\ x & \text{if } x \in X_1, y \in X_2, y = 0, \end{cases}$$

Then $(X; *, 0)$ is a BCI-algebra, and thus it is a Smarandache BCI-algebra.

Let $(X; *, 0)$ be a Smarandache BCI-algebra and let H be a subset of X such that $0 \in H$ and $|H| \geq 2$. Then H is called a *Smarandache subalgebra* of X if $(H; *, 0)$ is a Smarandache BCI-algebra. For example, consider a Smarandache BCI-algebra $X = \{0,1,2,3,4,5\}$ with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	3	2	3	3
1	1	0	3	2	3	3
2	2	2	0	3	0	0
3	3	3	2	0	2	2
4	4	2	1	3	0	1
5	5	2	1	3	1	0

Table 7

Then $H_1 = \{0,1,2,3\}$ is a Smarandache subalgebra of X. Any subalgebra of a Smarandache BCI-algebra X need not in general be a Smarandache subalgebra of X. For example, in the Smarandache BCI-algebra $X = \{0,1,2,3,4,5\}$ with the Table 7, the set $H_2 = \{0,2,3\}$ is a subalgebra of X which is not a Smarandache subalgebra of X. For a Smarandache BCI-algebra X, let H be a subalgebra of X. If H have a Cayley table of type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then H is a Smarandache subalgebra of X. If a BCI-algebra $(X; *, 0)$ contains a Smarandache subalgebra, then X is a Smarandache BCI-algebra. In fact, let H be a Smarandache subalgebra of X. Then there is a proper subset Q of H such that $0 \in Q$, $|Q| \geq 2$ and $(Q; *, 0)$ is a BCK-algebra. Since $H \subset X$, it follows that X is a Smarandache BCI-algebra. Let X be a Smarandache BCI-algebra. A nonempty subset I of X is called a *Smarandache ideal* of X related to Q if it satisfies:

(i) $0 \in I$,

(ii) $(\forall x \in Q)\ (\forall y \in I)\ (x * y \in I \Rightarrow x \in I)$, where Q is a BCK-algebra contained in X. If I is a Smarandache ideal of X related to every BCK-algebra contained in X, we simply say that I is a *Smarandache ideal* of X. Since X_+ is a maximal BCK-algebra contained in X, every subset I of a Smarandache BCI-algebra X containing X_+ is a Smarandache ideal of X. Let $X = \{0,1,2,3,4,5\}$ be a Smarandache BCI-algebra with the table 5. Then subsets $I = \{0,1,2\}$ and $J = \{0,1,3\}$ of X are Smarandache ideals of X related to a BCK-algebra $Q = \{0,1,2,3\}$ with respect to the operation $*$ on X. Let Q_1 and Q_2 be BCK-algebras contained in a Smarandache BCI-algebra X and $Q_1 \subset Q_2$. Then every Smarandache ideal of X related to Q_2 is a Smarandache ideal of X related to Q_1, but the converse is not true. For example, consider BCK-algebras $Q_1 = \{0,1,2\}$ and $Q_2 = \{0,1,2,3\}$ in a Smarandache BCI-algebra $X = \{0,1,2,3,4,5\}$ with the table 5. Then a subset $I = \{0,2,3\}$ is a Smarandache ideal of X related to Q_1, but not a Smarandache ideal of X related to Q_2 since $1 * 2 = 0 \in I$ and $2 \in I$ but $1 \notin I$. Thus we know that there exists a BCK-algebra Q contained in a Smarandache BCI-algebra X such that a Smarandache ideal of X related to Q is not an ideal of X.

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Y. B. JUN, DEPARTMENT OF MATHEMATICS EDUCATION, GYEONGSANG NATIONAL UNIVERSITY, CHINJU 660-701, KOREA

E-mail address: ybjun@gsnu.ac.kr jamjana@korea.com