Question 420 : some formulas related with pi

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Abstract: This note presents some formulas related with π .

Formulas

$$\frac{3\pi}{10}=\int_0^1\sqrt{-\frac{14}{3}+\sqrt[3]{\frac{18}{x}-\frac{143}{27}+2\sqrt{\frac{81}{x^2}-\frac{143}{3x}-\frac{100}{3}}}+\sqrt[3]{\frac{18}{x}-\frac{143}{27}-2\sqrt{\frac{81}{x^2}-\frac{143}{3x}-\frac{100}{3}}}}\,dx \quad (1)$$

$$\frac{3\pi}{10}=\int_0^1\cos^{-1}\sqrt{\frac{11x+\sqrt{25x^2+144x}}{72-48x}}\,dx \quad (2)$$

$$\frac{\pi}{5}=\int_0^1\sin^{-1}\sqrt{\frac{11x+\sqrt{25x^2+144x}}{72-48x}}\,dx \quad (3)$$

$$\pi=\frac{11}{3}-2\int_0^{1/2}\left(\frac{x}{3}+\sqrt[3]{\frac{x}{2}+\frac{x^3}{27}+\sqrt{\frac{x^2}{4}+\frac{x^4}{27}}}+\sqrt[3]{\frac{x}{2}+\frac{x^3}{27}-\sqrt{\frac{x^2}{4}+\frac{x^4}{27}}}\right)^2 dx \quad (4)$$

$$\pi=\frac{2}{3}+2\int_0^1\sqrt{-\frac{1}{3}+\sqrt[3]{\frac{1}{x}-\frac{1}{27}+\sqrt{\frac{1}{x^2}-\frac{2}{27x}}}+\sqrt[3]{\frac{1}{x}-\frac{1}{27}-\sqrt{\frac{1}{x^2}-\frac{2}{27x}}}}\,dx \quad (5)$$

$$\pi=\frac{14}{3}-\int_0^1\sqrt{x+\sqrt{8x+x^2}}\,dx \quad (6)$$

$$\pi=\frac{8}{3}+2\int_0^1\sqrt[4]{x+x\sqrt{x+x\sqrt{x+x\sqrt{x+\ldots}}}}\,dx \tag{7}$$

$$\pi=\frac{11}{3}-2\int_0^{1/2}\sqrt[3/2]{x+x\sqrt[3/2]{x+x\sqrt[3/2]{x+\ldots}}}\,dx \tag{8}$$

$$\pi=\frac{8}{3}+2\sqrt{2}\ln\left(1+\sqrt{2}\right)-4\int_0^{1/\sqrt{2}}\sinh^{-1}\left(\sqrt[8]{x^2+x^2\sqrt[4]{x^2+x^2\sqrt[4]{x^2+\ldots}}}\right)dx \tag{9}$$

$$\pi=\frac{8}{3}+2\sqrt{2}\ln\left(1+\sqrt{2}\right)-4\int_0^{1/\sqrt{2}}\cosh^{-1}\left(\sqrt{1+\sqrt{x\sqrt{1+\sqrt{x\sqrt{1+\ldots}}}}}\right)dx \tag{10}$$

$$\pi=\frac{2}{3}+2\int_0^1\sqrt[4]{\frac{2}{x+x\sqrt{\frac{2}{x+x\sqrt{\frac{2}{x+\ldots}}}}}}\,dx \tag{11}$$

$$\frac{1}{2\sqrt{2\pi}}\Gamma\left(\frac{1}{4}\right)^2-2\pi\sqrt{2\pi}\,\Gamma\left(\frac{1}{4}\right)^{-2}-1=$$
$$=\int_1^\infty\left(1-\sqrt{1-\left(\sqrt[3]{\sqrt{\frac{1}{27}+\frac{1}{x^2}}+\frac{1}{x}}-\sqrt[3]{\sqrt{\frac{1}{27}+\frac{1}{x^2}}-\frac{1}{x}}\right)^4}\right)dx \tag{12}$$

$$\pi=12\ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)-12\int_0^1\sinh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)dx \tag{13}$$

$$\pi=6\ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)-6\int_{1/\sqrt{2}}^1\cosh^{-1}\left(\frac{\sqrt{3}\,x}{\sqrt{1+x^2}}\right)dx \tag{14}$$

$$\pi F\left(\frac{1}{2},\frac{1}{2};1;\frac{3}{4}\right)=\int_0^\infty\left(\sqrt{2\sqrt{1+4x^{-2}}-1}-1\right)dx \tag{15}$$

$$\pi = \frac{9\sqrt{3}}{2} - 18\int_0^{\sqrt[3]{9}/4} \sqrt{-\frac{2}{\sqrt{3x}}\cos\left(\frac{4\pi}{3} + \frac{1}{3}\cos^{-1}\left(\frac{\sqrt{27x^3}}{2}\right)\right)}\, dx =$$

$$= \frac{9\sqrt{3}}{2} - \frac{18}{\sqrt[4]{3}}\int_0^{\sqrt[3]{9}/4} \sqrt{\frac{1}{\sqrt{x}}\left(\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{\sqrt{27x^3}}{2}\right)\right) - \sqrt{3}\sin\left(\frac{1}{3}\cos^{-1}\left(\frac{\sqrt{27x^3}}{2}\right)\right)\right)}\, dx \tag{16}$$

$$\pi = 6\sqrt{3} - 6\sqrt{6}\int_0^1 \frac{1}{\sqrt{1+\sqrt{1+48x^2}}}\, dx \tag{17}$$

$$\frac{\pi}{12} = \frac{1}{2} - \int_0^u \sqrt{-\frac{2}{\sqrt{3x}}\cos\left(\frac{4\pi}{3} + \frac{1}{3}\cos^{-1}\left(\frac{\sqrt{27x^3}}{2}\right)\right)}\, dx +$$

$$+ \int_{1/2}^u \sqrt{-\frac{2}{\sqrt{3x}}\cos\left(\frac{2\pi}{3} + \frac{1}{3}\cos^{-1}\left(\frac{\sqrt{27x^3}}{2}\right)\right)}\, dx \tag{18}$$

$$u = \frac{\sqrt[3]{4}}{3}$$

$$\frac{\pi^2}{8\sqrt{a^2-1}} = \int_0^\infty \frac{\tan^{-1}\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}\,(1+x^2)}\, dx \tag{19}$$

$$a = \sqrt{4-\sqrt{2}+2\sqrt{2-\sqrt{2}}}$$

$$\frac{3\pi}{32} - \frac{\pi}{16}\coth\pi = \sum_{n=1}^{\infty}\frac{1}{n^4+4}\int_0^1\sqrt{\frac{1-x}{n^4+4x}}dx = \int_0^1\left(\sum_{n=1}^{\infty}\frac{1}{n^4+4}\sqrt{\frac{1-x}{n^4+4x}}\right)dx \tag{20}$$

$$\frac{\pi}{4} - \frac{\pi}{4}\tanh\frac{\pi}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^4+1}\int_0^1\sqrt{\frac{1-x}{4n^4+1}}\, dx = \int_0^1\left(\sum_{n=1}^{\infty}\frac{1}{4n^4+1}\sqrt{\frac{1-x}{4n^4+x}}\right)dx \tag{21}$$

$$\frac{\pi}{\sqrt{6}} - \frac{\sqrt{3}}{\pi\sqrt[3]{4}}\Gamma\left(\frac{2}{3}\right)^3 = \int_0^{\sqrt{2/3}} \sin^{-1}\left(\frac{x^2-1}{3} + \sqrt[3]{p(x)+x\sqrt{q(x)}} + \sqrt[3]{p(x)-x\sqrt{q(x)}}\right)dx \tag{22}$$

$$p(x) = \frac{x^6}{27} + \frac{x^4}{18} + \frac{4x^2}{9} - \frac{1}{27},\; q(x) = \frac{x^6}{36} + \frac{x^4}{27} + \frac{5x^2}{27} - \frac{1}{27}$$

$$\pi = 4\sum_{n=0}^{\infty}\left(-\frac{d}{c}\right)^{n}\sum_{k=0}^{[n/2]}\binom{n-k}{k}\left(-\frac{c}{d^{2}}\right)^{k}\left(\frac{a}{c(2n+1)}-\frac{b}{c(2n+3)}\right) \tag{23}$$

$$a = 1136445, b = 219, c = 1428025, d = 57146$$

$$\pi = \frac{14}{3} - \frac{4\sqrt[4]{8}}{5}F\left(-\frac{1}{4},\frac{1}{4},\frac{5}{4};\frac{1}{2},\frac{9}{4};-\frac{1}{8}\right) - \frac{\sqrt[4]{2}}{7}F\left(\frac{1}{4},\frac{3}{4},\frac{7}{4};\frac{3}{2},\frac{11}{4};-\frac{1}{8}\right) \tag{24}$$

$$\frac{1}{12\sqrt{2\pi}}\Gamma\left(\frac{1}{4}\right)^{2} = \int_{0}^{\infty}\left(1-\sqrt[4]{\sqrt[3]{\sqrt{\frac{1}{4}+\frac{1}{27x^{6}}}+\frac{1}{2}}-\sqrt[3]{\sqrt{\frac{1}{4}+\frac{1}{27x^{6}}}-\frac{1}{2}}}\right)dx \tag{25}$$

$$\int_{u}^{\infty}\left(\sqrt[4]{\sqrt[3]{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{1}{27x^{6}}}}+\sqrt[3]{\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{1}{27x^{6}}}}}-1\right)dx = \\ = \frac{1}{12\sqrt{2\pi}}\Gamma\left(\frac{1}{4}\right)^{2} - \sqrt[6]{\frac{4}{27}}\left(\sqrt[6]{2}-1\right) - \frac{1}{3\sqrt{2}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-4n}}{4n+1} \tag{26}$$

$$u = \sqrt[6]{\frac{4}{27}}$$

$$\frac{\pi^{2}}{96} - \frac{1}{2}\left(\ln\left(1+\frac{2}{1+\sqrt{2}+\sqrt{3}}\right)\right)^{2} = \sum_{n=1}^{\infty}\frac{1}{n^{2}\binom{3n}{n}}\left(\frac{4}{14+9\sqrt{2}+8\sqrt{3}+5\sqrt{6}}\right)^{n} \tag{27}$$

$$\frac{\pi^{2}}{96} - \frac{1}{2}\left(\ln\left(2-\sqrt{2}\right)\right)^{2} = \sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^{2}\binom{3n}{n}\left(4+3\sqrt{2}\right)^{n}} \tag{28}$$

$$\frac{\pi}{3} + \frac{\ln 2}{3} = \frac{8}{9} + \int_{0}^{1}\int_{0}^{1}\left(\frac{1}{6}\sqrt[3]{f(x,y)} + \frac{2}{3}\frac{(xy)^{2}}{\sqrt[3]{f(x,y)}} + \frac{xy}{3}\right)^{2}\frac{1}{(1+x)^{3}}\,dxdy \tag{29}$$

$$f(x,y) = 108(1+x)^{2}y + 8(xy)^{3} + 12(1+x)y\sqrt{81(1+x)^{2}+12x^{3}y^{2}} \tag{30}$$

$$\frac{\pi}{3}+\frac{\ln 2}{3}=\frac{8}{9}+\int_0^1\int_0^1\left(\frac{1}{1+x}\sqrt[3/2]{\frac{y}{1+x}+\frac{xy}{1+x}\sqrt[3/2]{\frac{y}{1+x}+\frac{xy}{1+x}\sqrt[3/2]{\frac{y}{1+x}+\ldots}}}\right)dxdy \tag{31}$$

For $u=\dfrac{\sqrt{42+6\sqrt{17}}-\sqrt{17}-3}{4}$ we have

$$\frac{\pi}{24}+\frac{1}{4\sqrt{3}}\ln\left(\frac{1+u\sqrt{3}+u^2}{1-u\sqrt{3}+u^2}\right)=\int_0^u\frac{1}{1+x^6}dx=\sum_{n=0}^{\infty}\frac{(-1)^n u^{6n+1}}{6n+1} \tag{32}$$

For $u=\dfrac{\sqrt{42-6\sqrt{17}}-\sqrt{17}+3}{4}$ we have

$$\frac{\pi}{8}+\frac{1}{4\sqrt{3}}\ln\left(\frac{1+u\sqrt{3}+u^2}{1-u\sqrt{3}+u^2}\right)=\int_0^u\frac{1}{1+x^6}dx=\sum_{n=0}^{\infty}\frac{(-1)^n u^{6n+1}}{6n+1} \tag{33}$$

For $u=\dfrac{\sqrt{42-6\sqrt{17}}+\sqrt{17}-3}{4}$ we have

$$\frac{5\pi}{24}+\frac{1}{4\sqrt{3}}\ln\left(\frac{1+u\sqrt{3}+u^2}{1-u\sqrt{3}+u^2}\right)=\int_0^u\frac{1}{1+x^6}dx \tag{34}$$

For $u=\dfrac{\sqrt{42+6\sqrt{17}}+\sqrt{17}+3}{4}$ we have

$$\frac{7\pi}{24}+\frac{1}{4\sqrt{3}}\ln\left(\frac{1+u\sqrt{3}+u^2}{1-u\sqrt{3}+u^2}\right)=\int_0^u\frac{1}{1+x^6}dx \tag{35}$$

For $u=\dfrac{\sqrt{42-6\sqrt{17}}+\sqrt{17}-3}{4}$ we have

$$\pi=\frac{10\sqrt{3}\left(1+u^6\right)\ln\left(\dfrac{1+u\sqrt{3}+u^2}{1-u\sqrt{3}+u^2}\right)+24uF\left(1,1;\dfrac{11}{6};\dfrac{1}{1+u^6}\right)}{40u\left(1+u^6\right)^{5/6}F\left(\dfrac{1}{6},\dfrac{1}{6};\dfrac{1}{6};\dfrac{1}{1+u^6}\right)-25\left(1+u^6\right)} \tag{36}$$

For $u=\frac{\sqrt{42+6\sqrt{17}}+\sqrt{17}+3}{4}$ we have

$$\pi=\frac{10\sqrt{3}\left(1+u^6\right)\ln\left(\frac{1+u\sqrt{3}+u^2}{1-u\sqrt{3}+u^2}\right)+24uF\left(1,1;\frac{11}{6};\frac{1}{1+u^6}\right)}{40u\left(1+u^6\right)^{5/6}F\left(\frac{1}{6},\frac{1}{6};\frac{1}{6};\frac{1}{1+u^6}\right)-35\left(1+u^6\right)} \tag{37}$$

For $a=\frac{129991688+616283466i}{6103515625}, b=\frac{56169}{19531250}+\frac{369246}{9765625}i$, we have

$$\pi=4\sum_{n=1}^{\infty}\binom{2n-1}{n-1}\frac{1}{n}\operatorname{Im}\left(5a^n+7b^n\right) \tag{38}$$

A sequence related with π^4 :

$$x(n)=\sum_{k=1}^{n}(-1)^{k-1}\binom{4n}{4k}x(n-k) \quad ,x(0)=1,n\in\mathbb{N} \tag{39}$$

$$\frac{\pi^4}{4}=\lim_{n\to\infty}\left(\frac{x(n)}{x(n+1)}(4n+1)(4n+2)(4n+3)(4n+4)\right) \tag{40}$$

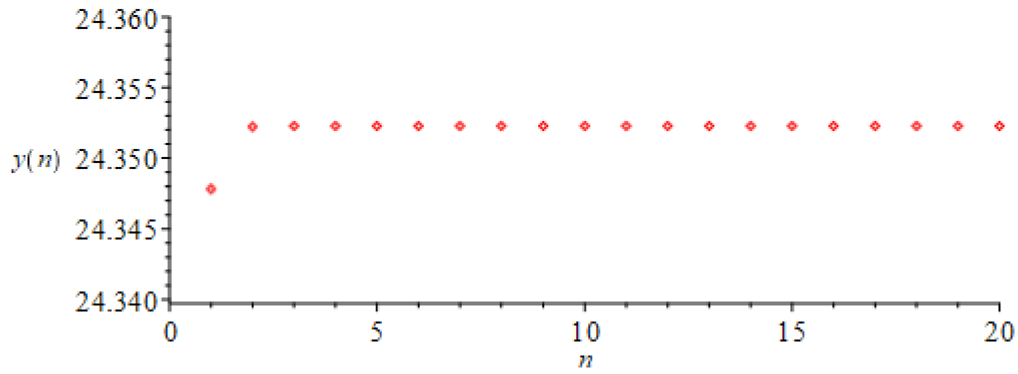

Fig. 1: $y(n)=\frac{x(n)}{x(n+1)}(4n+1)(4n+2)(4n+3)(4n+4)$, $n=0,1,2,3,\ldots$

Notation:

- ❖ $\Gamma(x)$ is the Gamma function.

- $F(a,b;c;x)$ is the hypergeometric function.
- $F(a,b,c;d,e;x)$ is the generalized hypergeometric function.
- $\mathrm{Im}(z)$ imaginary part of z .

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