A conjecture of existence of prime numbers on arithmetic progressions

Juan Moreno Borrallo

January 20, 2018

email: juan.morenoborrallo@gmail.com

Abstract

In this paper it is proposed a conjecture of existence of a prime number on an arithmetic progression, and demonstrated a particular case.

Main Conjecture. *Let two positive integer numbers be* a *and* b*,* $a \leq b$*. Then, it can be stated that at least one of the terms of the arithmetic progression* $S_{a,b} = \{ab+1, ab+2, ab+3, ..., ab+(b-1)\}$ *is a prime number.*

This conjecture is equivalent to the following:

Main Conjecture reformulation. *Let two positive integer numbers be* a *and* b*,* $a < b$*, such that* $\gcd(a,b) \geq \sqrt{a}$*. Then, it can be stated that exists at least one prime number* p *such that* $a < p < b$*.*

There are values of a and b for which the Conjecture is proved, or is similar with other conjectures or theorems. For instance, the Conjecture for $a = 1$ is Bertrand's Theorem, and for $a = b$ is somewhat similar to Oppermann's Conjecture.

In this paper, it is demonstrated the particular case $a = b$, when $(a+1)$ is a prime number.

1 Definitions and previous relationships

1.1 Theta function definition and relationships

Let us define the Theta function $\tau(n)$ as the number of different prime numbers that divide n. Thus,

$$\tau(n) = \# \{p \epsilon P \mid p \mid n\} \tag{1}$$

Let us define the number of prime numbers that divide ab as

$$\tau\left(b^2\right) = \#\left\{p\epsilon P \mid p \mid ab\right\} \tag{2}$$

Let us define the number of prime numbers that divide $(a+1)\,b$ as

$$\tau\left((a+1)b\right) = \#\left\{p\epsilon P \mid p \mid (a+1)b\right\} \tag{3}$$

It is easy to see that, if $a > 2$ and $(a+1)$ is a prime number, then a is a composite number of at least one factor, and thus

$$\tau\left((a+1)b\right) \leq \tau\left(ab\right) \tag{4}$$

Also, if $a > 2$ and a is a prime number, then $(a+1)$ is a composite number of at least one factor, and subsequently

$$\tau\left((a+1)b\right) \geq \tau\left(ab\right) \tag{5}$$

If both a and $(a{+}1)$ are composite numbers, we can't determine wether $\tau\left((a+1)b\right)$ or $\tau\left(ab\right)$ is greater. It is easy to check that the relationship $\tau\left((a+1)b\right) \geq \tau\left(ab\right)$ is more usual than $\tau\left((a+1)b\right) < \tau\left(ab\right)$, but we can find cases of this latest relationship (e.g., the case $a = 15$, and more generally every $a = 2^n - 1$ which is not a Mersenne prime number).

1.2 Pi function definition and relationships

Let us define the Pi function $\pi(n)$ as the number of prime numbers lesser or equal than n. Thus,

$$\pi(n) = \#\left\{p\epsilon P \mid p \leq n\right\} \tag{6}$$

Let us define the number of prime numbers less than ab and $(a+1)b$ as

$$\pi(b^2) = \#\left\{p\epsilon P \mid p \leq ab\right\} \tag{7}$$

$$\pi(b\,(b+1)) = \#\left\{p\epsilon P \mid p \leq (a+1)b\right\} \tag{8}$$

It is easy to see that, unconditionally,

$$\pi((a+1)b) \geq \pi(ab) \tag{9}$$

1.3 Phi function definition and relationships

Let us define the Euler's totient function as the number of positive integers lesser than n which are relative primes to n. Thus

$$\varphi(n) = \#\left\{k < n, k\epsilon\mathbb{N} \mid \gcd\left(k,n\right) = 1\right\} \tag{10}$$

Applying the multiplicative properties of $\varphi(n)$, we can state that

$$\varphi(ab) = \varphi(a)\varphi(b)\frac{d_1}{\varphi(d_1)} \tag{11}$$

Where $d_1 = \gcd(a, b)$. And

$$\varphi((a+1)b) = \varphi(a+1)\varphi(b)\frac{d_2}{\varphi(d_2)} \tag{12}$$

Where $d_2 = \gcd((a+1), b)$.

1.4 Delta function definition and relationships

Let us define the Delta function as the number of positive integers lesser than n which are relative primes to n and are composite.

$$\delta(n) = \#\{k < n, k\epsilon\mathbb{N} \mid \gcd(k, n) = 1,\ k\, composite\} \tag{13}$$

1.5 Relationships between theta function, pi function, phi function and delta function

According to the definitions and relationships established above, we can state the following equalities:

$$\varphi(ab) = \pi(ab) - \tau(ab) + \delta(ab) + 1 \tag{14}$$

$$\varphi((a+1)b) = \pi((a+1)b) - \tau((a+1)b) + \delta((a+1)b) + 1 \tag{15}$$

For the shake of clarity, number one is included as it is always considered a coprime number for every natural number and thus counted in Phi function, but is neither included in Delta function as it is not composite, nor in the Pi and Theta functions, as it is not considered a prime number.

2 Proof of Conjecture for the case $a = b$ and $(b+1)$ prime

2.1 Previous relationships

If we apply the definitions and relationships established in Section 1, then we have the following:

2.1.1 Theta function relationships

If $a = b$, as $\gcd(b, (b+1)) = 1$, then we can establish that

$$\tau(b(b+1)) \geq \tau\left(b^2\right) + 1 \tag{16}$$

2.1.2 Pi function relationships

$$\pi(b\,(b+1)) \geq \pi(b^2) \tag{17}$$

2.1.3 Phi function relationships

Applying the multiplicative properties of $\varphi(n)$, we can state that

$$\varphi(b^2) = b\varphi(b) \tag{18}$$

$$\varphi(b\,(b+1)) = \varphi(b)\varphi(b+1) \tag{19}$$

If $(b+1)$ is a prime number, then

$$\varphi(b+1) = b \tag{20}$$

And thus, substituting on (20) and comparing with (19)

$$\varphi(b\,(b+1)) = \varphi(b^2) \tag{21}$$

2.1.4 Delta function relationships

If $b > 2$ and $a = b$, then

$$\delta(b^2) \geq \delta((b+1)b) \tag{22}$$

Proof of (22)

1) As $\gcd(b,(b+1)) = 1$, there exist at least a subset of composite numbers $S_c = \{t_1(b+1),\, t_2(b+1), ..., t_{\varphi(b)-1}(b+1))\}$, where $t_1, t_2, ..., t_{\varphi(b)-1}$ are the coprime numbers to b lesser than b such that $t_n(b+1) < b^2$, whose elements are lesser than b^2 and counted in $\delta(b^2)$, but not counted in $\delta((b+1)b)$.

To be noted, $t_1 > 1$, and $t_{\varphi(b)-1} = (b-1)$, as $\gcd(b,(b+1)) = \gcd(b,(b-1)) = 1$, and $(b+1)(b-1) < b^2$. In total, there are $\varphi(b) - 1$ elements in the subset S_c.

2) In the arithmetic progression $S_{b^2} = \{b^2+1, b^2+2, b^2+3, ..., b^2+(b-1)\}$, there is a subset S_d of $\varphi(b)$ composite numbers coprime to b. However, as the numbers $b-1, b, b+1, b+2$ can not be coprime two to two at the same time, the number $b^2 + (b-2) = (b-1)(b+2)$ is not coprime to b and $b+1$ at the same time, and subsequently is not included in S_d. Thus, there are at most $\varphi(b) - 1$ elements in the subset S_c. This numbers are counted in $\delta((a+1)b)$ but not counted in $\delta(ab)$.

3) Subsequently, as $\parallel S_c \parallel \geq \parallel S_d \parallel$, then $\delta(b^2) \geq \delta((b+1)b)$.

2.1.5 Relationships between theta function, pi function, phi function and delta function

According to the definitions and relationships established above, we can state the following equalities:

$$\varphi(b^2) = \pi(b^2) - \tau\left(b^2\right) + \delta(b^2) + 1 \tag{23}$$

$$\varphi(b\,(b+1)) = \pi(b\,(b+1)) - \tau\,(b\,(b+1)) + \delta(b\,(b+1)) + 1 \tag{24}$$

For the shake of clarity, number one is included as it is always considered a coprime number for every natural number and thus counted in Phi function, but is neither included in Delta function as it is not composite, nor in the Pi and Theta functions, as it is not considered a prime number.

2.2 Proof development

Let us assume that

$$\pi(b\,(b+1)) = \pi(b^2) \tag{25}$$

For the case $(b{+}1)$ prime, as stated in (16),

$$\tau\,(b\,(b+1)) = \tau\left(b^2\right) + 1$$

If (25) holds, and substituting by (23) and (24) on (21), then it can be stated that

$$\varphi(b^2) = \varphi(b\,(b+1))$$

$$\left(\pi(b^2) - \tau\left(b^2\right) + \delta(b^2) + 1\right) = \left(\pi(b\,(b+1)) - \left(\tau\left(b^2\right) + 1\right) + \delta(b\,(b+1)) + 1\right) \tag{26}$$

Operating under the assumption, we get that

$$\delta(b^2) = \delta(b\,(b+1)) - 1 \tag{27}$$

However, according to (22), if $b > 2$ and $a = b$, then

$$\delta(b^2) \geq \delta(b\,(b+1))$$

Thus, (27) implies a contradiction with (22). The only way to avoid this contradiction is to consider that, for the case $b > 2$ and $(b+1)$ prime,

$$\pi(b\,(b+1)) \geq \pi(b^2) + 1 \tag{28}$$

It is easy to verify that the Conjecture holds for every $b \leq 2$ and $(b+1)$ prime.

Subsequently, the Conjecture is proved for the case $a = b$, $(b+1)$ prime, and it can be established the following theorem:

Theorem. *Let it be a positive integer n such that $n+1$ is a prime number. Then, it can be stated that at least one of the terms of the arithmetic progression $S_{n^2} = \left\{n^2+1, n^2+2, n^2+3, ..., n^2+(n-1)\right\}$is a prime number.*