The Positive Integer Solutions of Equation $Ax^m+By^n=Cz^k$

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Abstract: In this paper for equation $Ax^m+By^n=Cz^k$, where $m,n,k > 2$, $x,y,z > 1$, $A,B,C \geqslant 1$ and $\gcd(Ax,By,Cz)=1$, the author proved there are no positive integer solutions for this equation using "Order reducing method for equations" that the author invented for solving high order equations, in which let the equation become two equations, through comparing the two roots to prove there are no positive integer solutions for this equation.

1. Some Relevant Theorems

There are some theorems for proving or need to be known. ***All symbols in this paper represent positive integers unless stated they are not.***

Theorem 1.1. In equation

$$\begin{cases} Ax^m + By^n = Cz^k \\ x, y, z > 1 \\ m, n, k > 2 \\ A, B, C \geq 1 \\ \gcd(Ax, By, Cz) = 1 \end{cases} \tag{1-1}$$

Ax, By, Cz meet

$$\gcd(Ax, By) = \gcd(Ax, Cz) = \gcd(By, Cz) = 1.$$

Proof: Since

$$Ax^m + By^n = Cz^k ,$$

if $\gcd(Ax, By) > 1$ then

$$\gcd(Ax, By) \times \left(x_1^{m-1} x_2 + y_1^{n-1} y_2\right) = Cz^k$$

which means $\gcd(Ax, By, Cz) > 1$ that contradicts against $\gcd(Ax, By, Cz) = 1$ in equation (1-1) since the right side of Cz must contain the factor of $\gcd(Ax, By) > 1$. So we have $\gcd(Ax, By) = 1$ and using the same way we can also prove $\gcd(Ax, Cz) = \gcd(By, Cz) = 1$.

Theorem 1.2. There are no positive integer solutions for equation (1-1) when

$$(x = y) \neq z \text{, or } (x = z) \neq y \text{, or } (z = y) \neq x .$$

Proof: When

$$(x = y) \neq z,$$

from equation (1-1) we have

$$Ax^m + Bx^n = Cz^k,$$

let $m \geq n$, we get

$$(Ax^{m-n} + B)x^n = Cz^k$$

which means

$$\gcd(x, Cz) = x > 1$$

that contradicts against **Theorem 1.1** in which $\gcd(Ax, Cz) = 1$. Using the same ways we can also prove $(x = z) \neq y$ and $(z = y) \neq x$. So there are no positive integer solutions for equation (1-1) when $(x = y) \neq z$, or $(x = z) \neq y$, or $(z = y) \neq x$.

Theorem 1.3. Function $f(x) = D^U$ and $g(x) = D^U + E^V$ are all monotonically increasing "Convex functions", where D, E are all positive real numbers and U, V are real numbers.

Proof: Since monotonically increasing "Convex function" meets

$$f'(x) = \frac{df(x)}{dx} > 0,$$

$$f''(x) = \frac{d^2 f(x)}{dx^2} > 0,$$

for $f(x) = D^U$ and $g(x) = D^U + E^V$, we have

$$f'(x) = D^U \ln D > 0,$$
$$f''(x) = D^U \ln^2 D > 0,$$

$$g'(x) = D^U \ln D + E^V \ln E > 0,$$
$$g''(x) = D^U \ln^2 D + E^V \ln^2 E > 0,$$

so $f(x) = D^U$ and $g(x) = D^U + E^V$ are all monotonically increasing "Convex functions".

Theorem 1.4. Given A, B, C, x, y, z for equation (1-1) as showed in **Figure 1-1**, there is only one intersection for $Ax^{k+r_1} + By^{k+r_2} = Cz^k$, in which r_1, r_2 are finite integers. Let M, N, K

be the solution of equation (1-1). In this paper we will use $Ax^m + By^n$ in place of $Ax^{k+r_1} + By^{k+r_2}$ since they are of the same values but just at different places of w.

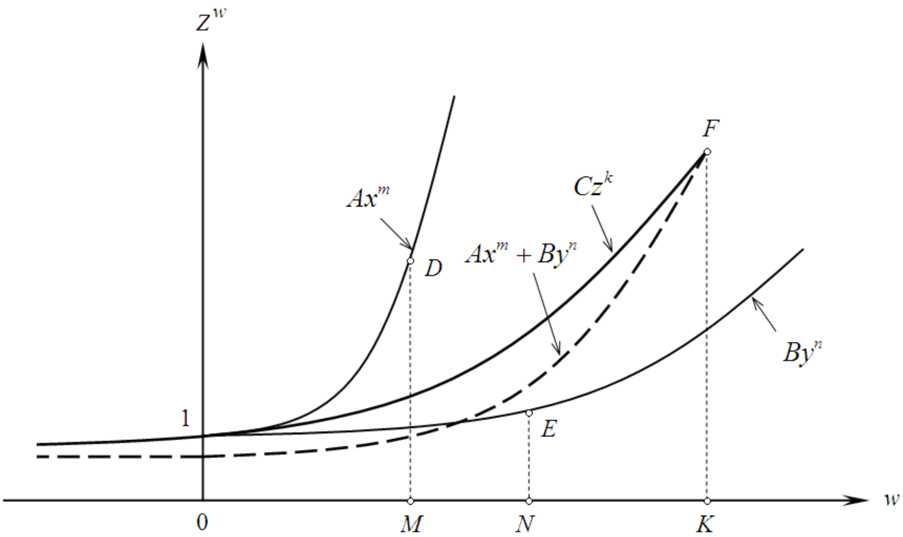

Figure 1-1 *Graph for* $Ax^m + By^n = Cz^k$

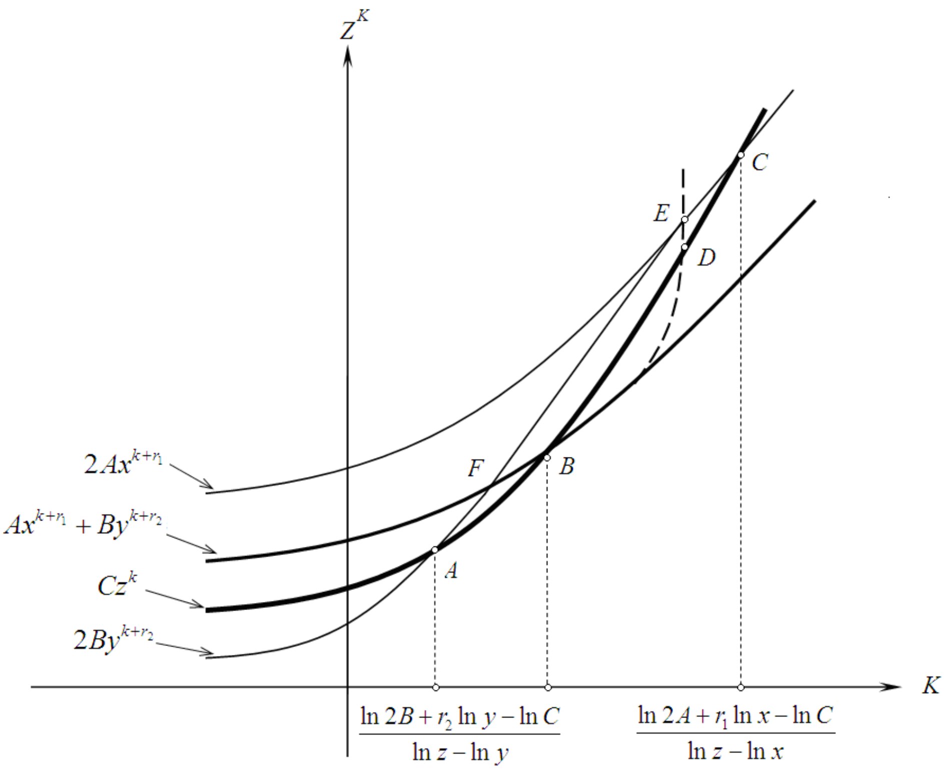

Figure 1-2 *Graph for* $Ax^{k+r_1} + By^{k+r_2} = Cz^k$ *and* $2Ax^{k+r_1}, 2By^{k+r_2}$

Proof: In **Figure 1-1** we can see clearly that since M, N, K is the solution of $x^m + y^n = z^k$, so we have $\left(DM = Ax^M\right)+\left(EN = By^N\right)=\left(Cz^K = FK\right)$, and by moving DM, EN to the same line of FK, we get the curve of $Ax^{k+r_1}+By^{k+r_2}$, the horizontal axis stands for the exponent of z. Because $Ax^{k+r_1}+By^{k+r_2}$, Cz^k are all monotonically increasing "Convex functions" , they have at most two intersections, but one of them is at $w \to -\infty$, this can be explained by **Figure 1-2**. Obviously we can see curve $Ax^{k+r_1}+By^{k+r_2}$ is "Between" curves $2Ax^{k+r_1}, 2By^{k+r_2}$, which means point *A* is the intersection of $2Ay^{k+r_2}, Cz^k$, and point *A* is the only intersection since

$$\left(2By^{k+r_2} = Cz^k\right) \Rightarrow \left[\ln 2B + \left(k + r_2\right)\ln y = \ln C + k\ln z\right] \Rightarrow \left(k = \frac{\ln 2B + r_2 \ln y - \ln C}{\ln z - \ln y}\right),$$

in which k is the only solution (*since* r_1, r_2 *are finite integers*) when given z, y, r_2, A, B, C.

Point *C* is the intersection of $2Ax^{k+r_1}, Cz^k$ and point *C* is the only intersection since

$$\left(2Ax^{k+r_1} = Cz^k\right) \Rightarrow \left[\ln 2A + \left(k + r_1\right)\ln x = \ln C + k\ln z\right] \Rightarrow \left(k = \frac{\ln 2A + r_1 \ln x - \ln C}{\ln z - \ln x}\right),$$

in which k is the only solution (*since* r_1, r_2 *are finite integers*) when given z, y, r_1, A, B, C.

Point *B* is the intersection of $Ax^{k+r_1}+By^{k+r_2}, Cz^k$. If there exists third intersection of curves $Ax^{k+r_1}+By^{k+r_2}, Cz^k$ which is point *D*, then the curve *DE* will intersect $2Ax^{k+r_1}$ at point *E*, that means

$$Ax^{k+r_1}+By^{k+r_2} = 2Ax^{k+r_1}$$

and

$$By^{k+r_2} = Ax^{k+r_1},$$

in which

$$\left[\ln B + \left(k + r_2\right)\ln y = \ln A + \left(k + r_1\right)\ln x\right] \Rightarrow \left(k = \frac{\ln A + r_1 \ln x - r_2 \ln y - \ln B}{\ln y - \ln x}\right),$$

point *E* is also the intersection of $2Ax^{k+r_1}, 2By^{k+r_2}$, which means curve $2By^{k+r_2}$ intersects $Ax^{k+r_1}+By^{k+r_2}$ twice times and have two intersections that are points *F*, *E*, this is a contradiction since curves $Ax^{k+r_1} + By^{k+r_2}$ and $2By^{k+r_2}$ have only one intersection. Using the same way we have the same conclusion when exchanging curves $Ax^{k+r_1} + By^{k+r_2}, Cz^k$ or $2Ax^{k+r_1}, 2By^{k+r_2}$ in **Figure 1-2**.

So we have the conclusion of point E is not existed and there is only one intersection when M, N, K is a solution of equation (1-1) for any given A, B, C, x, y, z.

Theorem 1.5. In **Figure 1-3**, in which

$$Ax^{m-i} + By^{n-i} < Cz^{k-i},$$

x, y, z of equation (1-1) meet

$$\frac{Ax^{m-1} + By^{n-1} - Cz^{k-1}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} < \frac{1}{2}.$$

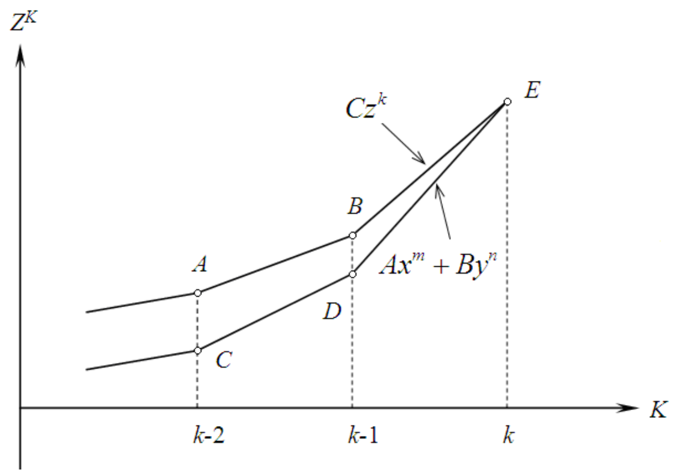

Figure 1-3 *Graph for* $Ax^m + By^n = Cz^k$ *when* $Ax^{m-i} + By^{n-i} < Cz^{k-i}$

Proof: The slope of CD, DE are

$$S_{CD} = \frac{(Ax^{m-1} + By^{n-1}) - (Ax^{m-2} + By^{n-2})}{(k-1) - (k-2)} = A(x-1)x^{m-2} + B(y-1)y^{n-2},$$

$$S_{DE} = \frac{(Ax^m + By^n) - (Ax^{m-1} + By^{n-1})}{k - (k-1)} = A(x-1)x^{m-1} + B(y-1)y^{n-1},$$

and

$$S_{DE} > S_{CD},$$

since $x > 1, y > 1$. The slope of AB, BE are

$$S_{AB} = \frac{Cz^{k-1} - Cz^{k-2}}{(k-1)-(k-2)} = C(z-1)z^{k-2},$$

$$S_{BE} = \frac{Cz^{k} - Cz^{k-1}}{k-(k-1)} = C(z-1)z^{k-1},$$

and

$$S_{BE} > S_{AB},$$

since $z > 1$, so we have

$$S_{BE} + S_{DE} > S_{AB} + S_{CD}$$

and

$$A(x-1)x^{m-1} + B(y-1)y^{n-1} + C(z-1)z^{k-1} > A(x-1)x^{m-2} + B(y-1)y^{n-2} + C(z-1)z^{k-2},$$

we get

$$\frac{Ax^{m-1} + By^{n-1} - Cz^{k-1}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} < \frac{C(z-1)\left(z^{k-1} - z^{k-2}\right)}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} + \frac{1}{2}$$

where

$$\begin{cases} C(z-1)\left(z^{k-1} - z^{k-2}\right) > 0 \\ Ax^{m-2} + By^{n-2} - Cz^{k-2} < 0 \end{cases},$$

so we have

$$\frac{Ax^{m-1} + By^{n-1} - Cz^{k-1}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} < \frac{1}{2}.$$

Theorem 1.6. In **Figure 1-4**, in which

$$Ax^{m-i} + By^{n-i} > Cz^{k-i},$$

x, y, z of equation (1-1) meet

$$\frac{Ax^{m-1} + By^{n-1} - Cz^{k-1}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} \le 1.$$

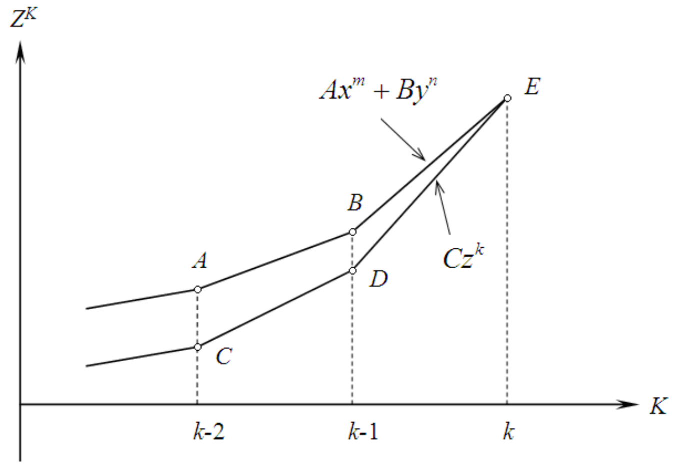

Figure 1-4 *Graph for* $Ax^m + By^n = Cz^k$ *when* $Ax^{m-i} + By^{n-i} > Cz^{k-i}$

Proof: Obviously the meaning of $\dfrac{Ax^{m-1} + By^{n-1} - Cz^{k-1}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} \le 1$ is the slope of AB is not greater than that of CD. If $\dfrac{Ax^{m-1} + By^{n-1} - Cz^{k-1}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} > 1$ then are three cases have to be considered. From **Theorem 1.3** we have already known that $Ax^m + By^n, Cz^k$ are all monotonically increasing "Convex functions".

The first case (**Case I**) is there is a positive real number $0 < r < 1$ for $k - r$ between $k - 1$ and k whose slope equals to that of AB which means

$$Ax^{m-1} + By^{n-1} - Ax^{m-2} - By^{n-2} = \frac{Cz^{k-r} - Cz^{k-1}}{1-r} = \frac{C\left(z^{1-r} - 1\right)z^{k-1}}{1-r}$$

that can be explained by **Figure 1-5** where $AB \mathbin{//} DF$.

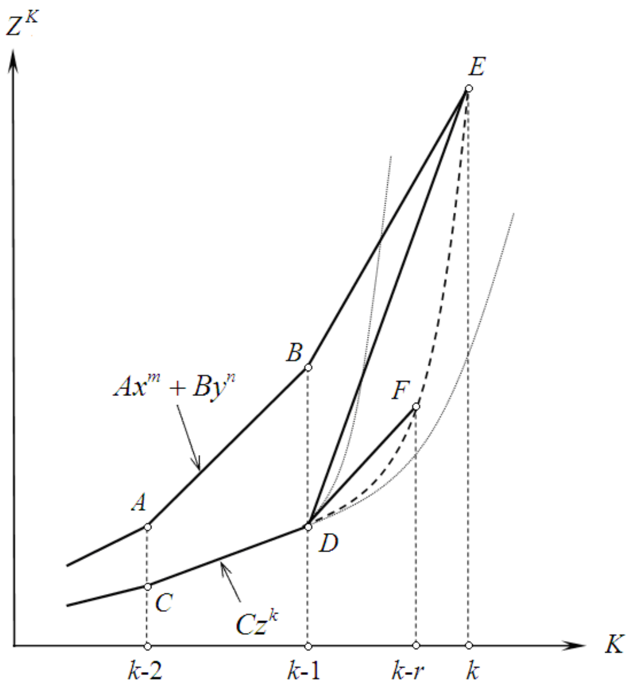

Figure 1-5 *Graph of* $Ax^m + By^n = Cz^k$ *when* $\dfrac{Ax^{m-1} + By^{n-1} - Cz^{k-1}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} \geq 1$,

point F *is between* k-1 *and* k *for* ***Case I***

The second case (**Case II**) is there is a positive real number $0 < r < 1$ for $k - r - 1$ between $k - 1$ and $k - 2$ whose slope equals to that of AB which means

$$Ax^{m-1} + By^{n-1} - Ax^{m-2} - By^{n-2} = \frac{Cz^{k-1} - Cz^{k-r-1}}{r} = \frac{C\left(1 - z^{-r}\right)z^{k-1}}{r}$$

that can be explained by **Figure 1-6** where $AB \,//\, DF$.

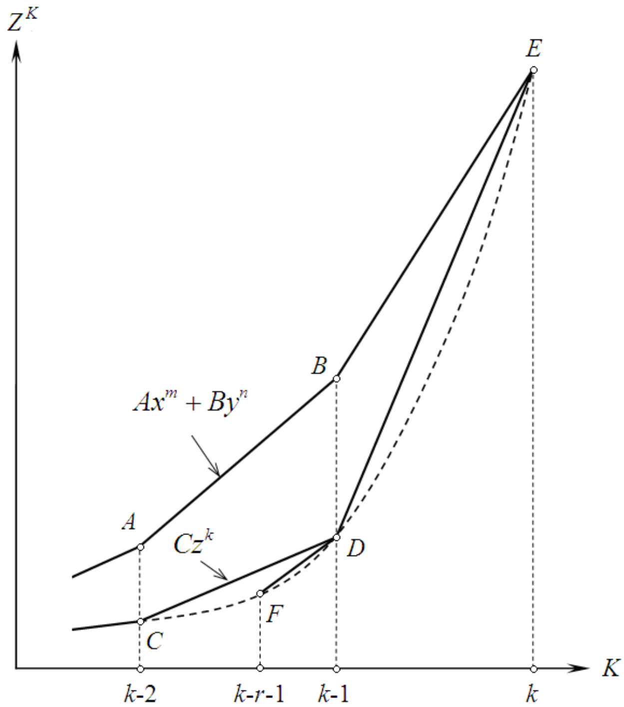

Figure 1-6 *Graph of* $Ax^m + By^n = Cz^k$ *when* $\dfrac{Ax^{m-1} + By^{n-1} - Cz^{k-1}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} \geq 1$,

point F *is between* k-2 *and* k-1 *for* ***Case II***

The third case (**Case III**) is there is a tangent line of curve Cz^k at D that is $D'DF$ whose slope equals to that of AB which means

$$Ax^{m-1} + By^{n-1} - Ax^{m-2} - By^{n-2} = \frac{Cdz^K}{dK}\Big|_{K=k-1}$$

that can be explained by **Figure 1-7** where $AB \,//\, D'DF$.

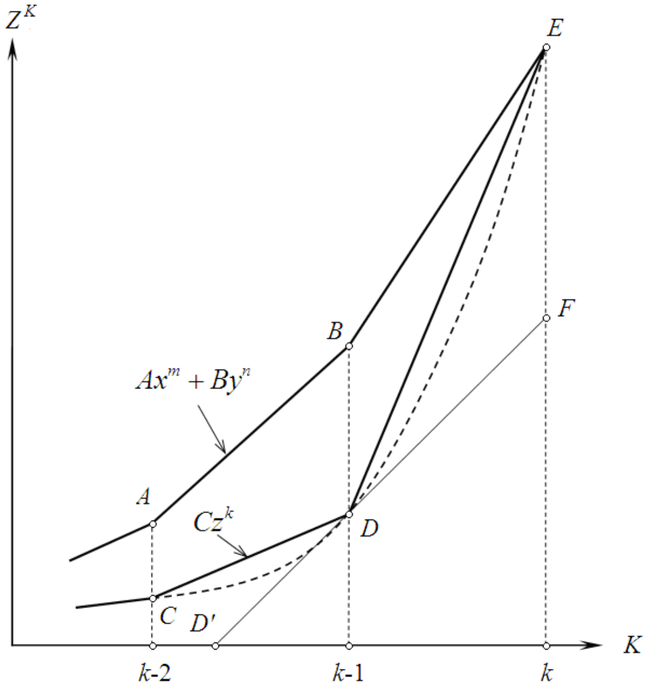

Figure 1-7 *Graph of* $Ax^m + By^n = Cz^k$ *when* $\dfrac{Ax^{m-1} + By^{n-1} - Cz^{k-1}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} \ge 1$,

$D'DF$ *is a tangent line of curve* Cz^k *for* ***Case III***

Case I : In **Figure 1-5** we have

$$Ax^{m-1} + By^{n-1} - Ax^{m-2} - By^{n-2} = C\left(\frac{z^{1-r}-1}{1-r}\right)z^{k-1},$$

and

$$Ax^{m-1} + By^{n-1} - Cz^{k-1} - Ax^{m-2} - By^{n-2} = C\left(\frac{z^{1-r}-1}{1-r}\right)z^{k-1} - Cz^{k-1}$$
$$= C\left(\frac{z^{1-r}+r-2}{1-r}\right)z^{k-1} \quad . \tag{1-2}$$

If we treat r as constant then $f(z) = \dfrac{z^{1-r}+r-2}{1-r}$ is a "Monotonically increasing function"; if we treat z as constant then $f(r) = \dfrac{z^{1-r}+r-2}{1-r}$ is a "Monotonically decreasing function" that is because:

$$f'(r)=\frac{d\left(\dfrac{z^{1-r}+r-2}{1-r}\right)}{dr}=\frac{\left(-z^{1-r}\ln z+1\right)(1-r)+z^{1-r}+r-2}{(1-r)^2}.$$

$$=\frac{-z^{1-r}\ln z(1-r)+z^{1-r}-1}{(1-r)^2}=\frac{[(r-1)\ln z+1]z^{1-r}-1}{(1-r)^2}$$

For function

$$g(z)=\frac{[(r-1)\ln z+1]z^{1-r}-1}{(1-r)^2},$$

it is a "Monotonically decreasing function" since

$$g'(z)=\frac{d\left\{\dfrac{[(r-1)\ln z+1]z^{1-r}-1}{(1-r)^2}\right\}}{dz}=\frac{\dfrac{(r-1)}{z}z^{1-r}+(1-r)z^{-r}[(r-1)\ln z+1]}{(1-r)^2}$$

$$=-z^{-r}\ln z<0.$$

For function

$$g(r)=\frac{[(r-1)\ln z+1]z^{1-r}-1}{(1-r)^2},$$

we give the plot of it in **Figure 1-8**, in which it shows that $g(r)\neq 0$ and $g(r)<0$ (*we have to say because we can not solve "Exponent equation" where the "Exponent" is unknown number, so the solutions have to be found in numerical way, which is just "Function plot" does*). When $0<r<1$ the value of $g(r)$ is less than 0, since $f'(r)=g(r)<0$ and $g(z)$ is a "Monotonically decreasing function", so $f(r)$ is a "Monotonically decreasing function".

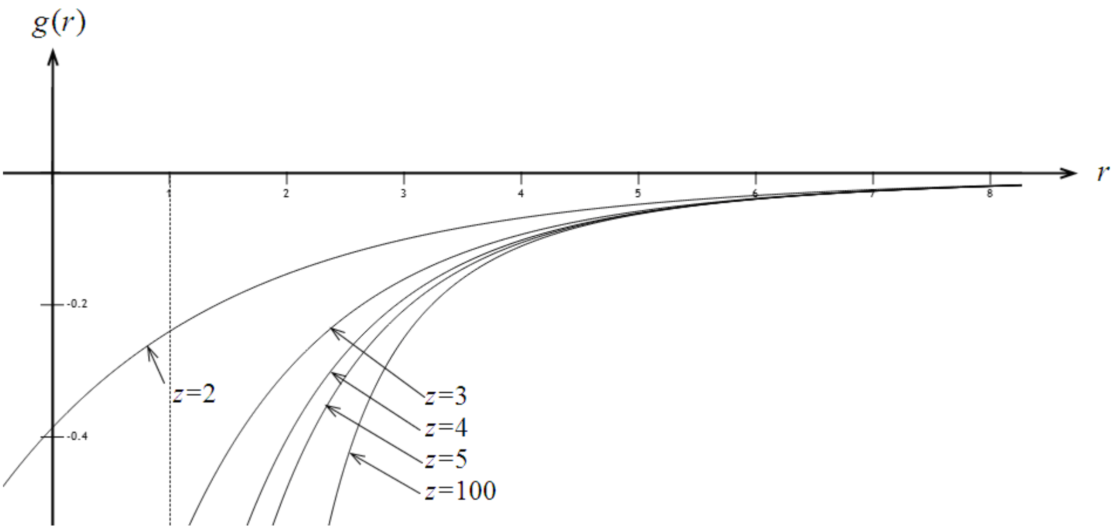

Figure 1-8 *Graph of* $g(r)=\dfrac{[(r-1)\ln z+1]z^{1-r}-1}{(1-r)^2}$ *when* $z=2,3,4,5,100$

From (1-2) we can see clearly that if z (a positive number) increases then the left side decreases and the right side also decreases. The minimum value for the right side is

$$\lim_{r\to 1} C\left(\frac{z^{1-r}+r-2}{1-r}\right)z^{k-1} = \lim_{r\to 1} C\left[\frac{\frac{d\left(z^{1-r}+r-2\right)}{dr}}{\frac{d(1-r)}{dr}}\right]z^{k-1} = \lim_{r\to 1} C\left(\frac{-z^{1-r}\ln z+1}{-1}\right)z^{k-1},$$

$$= \lim_{r\to 1} C\left(z^{1-r}\ln z-1\right)z^{k-1} = C(\ln z-1)z^{k-1}$$

since

$$\begin{cases}\lim_{r\to 1}\left(z^{1-r}+r-2\right)=0\\ \lim_{r\to 1}(1-r)=0\end{cases}.$$

From **Theorem 1.8** we know $z \geq 5$, and since $C \geq 1$, so we get

$$\left[\lim_{r\to 1} C\left(\frac{z^{1-r}+r-2}{1-r}\right)z^{k-1} = C(\ln z-1)z^{k-1}\right] \geq (\ln 5-1)\times 5^2 > 15.$$

From (1-2) we have

$$\left(Ax^{m-1}+By^{n-1}-Cz^{k-1}\right)-\left(Ax^{m-2}+By^{n-2}-Cz^{k-2}\right)=C\left(\frac{z^{1-r}+r-2}{1-r}\right)z^{k-1}+Cz^{k-2},$$

where both sides plus Cz^{k-2}. In **Figure 1-5** we know

$$Ax^{m-1}+By^{n-1}-Cz^{k-1}=BD,$$

$$Ax^{m-2}+By^{n-2}-Cz^{k-2}=AC,$$

there must exist a situation in **Figure 1-5** when we increase z that causes $BD \to AC, BD > AC, r < 1,$ so the left side is almost 0 but the right side is bigger than $15 + Cz^{k-2}$, that is a contradiction which means there are no positive integer solutions of equation (1-1) at **Case I.**

Case II : In **Figure 1-6** we have

$$Ax^{m-1}+By^{n-1}-Ax^{m-2}-By^{n-2}=\frac{C\left(1-z^{-r}\right)z^{k-1}}{r},$$

and

$$Ax^{m-1}+By^{n-1}-Cz^{k-1}-Ax^{m-2}-By^{n-2}=C\left(\frac{1-z^{-r}}{r}\right)z^{k-1}-Cz^{k-1}$$
$$=C\left(\frac{1-z^{-r}-r}{r}\right)z^{k-1} \quad . \qquad (1\text{-}3)$$

For function $f(z)=\frac{1-z^{-r}-r}{r}$, it is clear that $f(z)$ is a “Monotonically increasing function” since z^{-r} is a “Monotonically decreasing function”. For function $f(r)=\frac{1-z^{-r}-r}{r}$, it is a “Monotonically decreasing function”, that is because:

$$f'(r)=\frac{d\left(\frac{1-z^{-r}-r}{r}\right)}{dr}=\frac{rz^{-r}\ln z-r-\left(1-z^{-r}-r\right)}{r^2}=\frac{\left(r\ln z+1\right)z^{-r}-1}{r^2}.$$

For function

$$g(z)=\frac{\left(r\ln z+1\right)z^{-r}-1}{r^2},$$

it is a “Monotonically decreasing function” since

$$g'(z)=\frac{d\left[\frac{\left(r\ln z+1\right)z^{-r}-1}{r^2}\right]}{dz}=\frac{\left[\frac{r}{z}-r\left(r\ln z+1\right)\right]z^{-r}}{r^2}<0,$$

in which from **Theorem 1.8** we know $z\geq 5$, so we have $\frac{r}{z}-r\left(r\ln z+1\right)<0$ where $\frac{r}{z}<r$ and $r^2\ln z>0$.

For function $g(r)=\frac{\left(r\ln z+1\right)z^{-r}-1}{r^2}$, we plot the graph of it in **Figure 1-9**, in which it shows that $g(r)\neq 0$ and $g(r)<0$. When $0<r<1$ the value of $g(r)$ is less than 0, since $f'(r)=g(r)<0$ and $g(z)$ is a “Monotonically decreasing function”, so $f(r)$ is a “Monotonically decreasing function”.

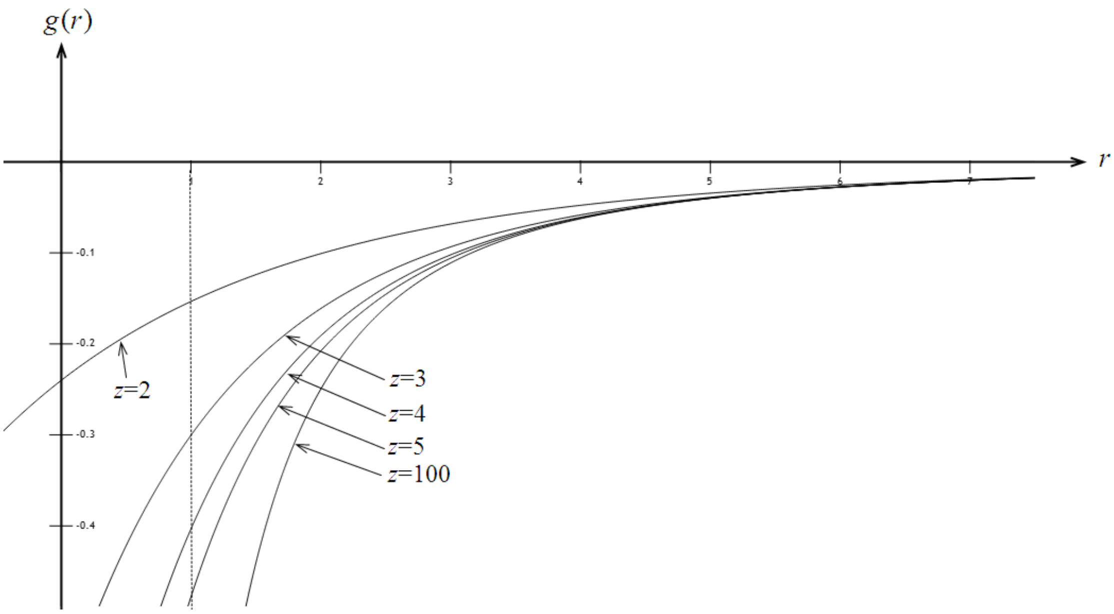

Figure 1-9 *Graph of* $g(r) = \frac{(r\ln z + 1)z^{-r} - 1}{r^2}$ *when* $z = 2,3,4,5,100$

From **Figure 1-6** we know if z (a positive real number) increases then r increases too. From (1-3) we have

$$\left(Ax^{m-1} + By^{n-1} - Cz^{k-1}\right) - \left(Ax^{m-2} + By^{n-2} - Cz^{k-2}\right) = C\left(\frac{1 - z^{-r} - r}{r}\right)z^{k-1} + Cz^{k-2}$$

where both sides plus Cz^{n-2}, in **Figure 1-6** we know

$$Ax^{m-1} + By^{n-1} - Cz^{k-1} = BD,$$

$$Ax^{m-2} + By^{n-2} - Cz^{k-2} = AC\,,$$

there must exist a situation when we increase z (a positive real number) that causes

$$BD \to AC, BD > AC, r \to 1, r < 1,$$

so the left side is

$$\left(Ax^{m-1} + By^{n-1} - Cz^{k-1}\right) - \left(Ax^{m-2} + By^{n-2} - Cz^{k-2}\right) = 0_+ > 0\,,$$

when $r = 1$ the right side is

$$C\left[\left(\frac{1 - z^{-r} - r}{r}\right)z^{k-1} + z^{k-2}\right] = \left(-z^{k-1-r} + z^{k-2}\right) = 0\,,$$

since $f(r) = \frac{1 - z^{-r} - r}{r}$ is a "Monotonically decreasing function", so when $r < 1$, the right side is greater than 0, we can not have contradiction as **Case I** does. But **Case II** is still impossible, since in **Figure 1-3**, it is obvious that

$$\angle CDE = 360^0 - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-1}}\right),$$

$$\angle CDE < \angle ADE,$$

from **Theorem 1.8** we know $z \ge 5$ there are no positive integer solutions for equation (1-1), when $n = 3$ (*which is the worse case*) we have

$$\angle CDE = 360^0 - \arctan\left(\frac{z^k - z^{k-1}}{1}\right) - \arctan\left(\frac{1}{z^{k-1} - z^{k-2}}\right)$$
$$= 360^0 - \arctan(5^3 - 5^2) - \arctan\left(\frac{1}{5^2 - 5}\right) > 177.7^0,$$

and

$$\angle ADE > \angle CDE > 177.7^0,$$

which means $\angle ADE, \angle CDE \to 180^0$ with $z > 100, n = 3$, and ADE, CDE are almost lines that lead to the result of $BD < AC$, so this is a contradiction which means there are no positive integer solutions of equation (1-1) at **Case II**.

Case III : In **Figure 1-7** we have

$$Ax^{m-1} + By^{n-1} - Ax^{m-2} - By^{n-2} = \frac{Cdz^K}{dK}\Big|_{K=k-1} = Cz^{k-1}\ln z,$$

and

$$Ax^{m-1} + By^{n-1} - Cz^{k-1} = Cz^{k-1}\ln z - Cz^{k-1} + Ax^{m-2} + By^{n-2}$$
$$= C(\ln z - 1)z^{k-1} + Ax^{m-2} + By^{n-2},$$

that is impossible since for any positive integer solutions of equation (1-1) when z increases then the left side is becoming smaller but the right side is becoming bigger(*since from **Theorem 1.8** we know* $z \ge 5$, *so* $(\ln z - 1) > 0$) which is a contradiction, so there are no positive integer solutions of equation (1-1) at **Case III**.

So from **Case I**, **Case II** and **Case III** we have the conclusion of $\frac{Ax^{m-1} + By^{n-1} - Cz^{k-1}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} > 1$ is impossible and $\frac{Ax^{m-1} + By^{n-1} - Cz^{k-1}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} \le 1$.

Using the same way we can prove

$$\frac{Ax^{m-j} + By^{n-j} - Cz^{k-j}}{Ax^{m-j-1} + By^{n-j-1} - Cz^{k-j-1}} \le 1,$$

where j is a positive integer, this can be explained by **Figure 1-10** in which $w = k - j$, $w > 2,\ AC \geq BD$.

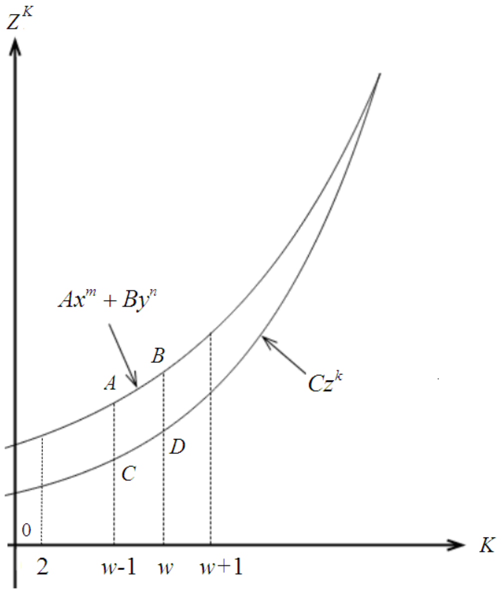

Figure 1-10 *Graph of* $Ax^m + By^n = Cz^k$ *when* $\dfrac{Ax^{m-j} + By^{n-j} - Cz^{k-j}}{Ax^{m-j-1} + By^{n-j-1} - Cz^{k-j-1}} \leq 1$

where j *is a positive integer and* $w = k - j$

Theorem 1.7. If $Ax^{m-2} + By^{n-2} > Cz^{k-2}$ then x, y, z of equation (1-1) meet

$$z > y$$

when

$$x > y.$$

Proof: Since

$$Ax^{m-2} + By^{n-2} > Cz^{k-2},$$

so we have

$$A\left(\frac{z}{x}\right)^2 x^m + B\left(\frac{z}{y}\right)^2 y^n > Cz^k.$$

If $\left(\dfrac{z}{x}\right)^2 < 1, \left(\dfrac{z}{y}\right)^2 < 1$ then

$$A\left(\frac{z}{x}\right)^2 x^m + B\left(\frac{z}{y}\right)^2 y^n < Cz^k$$

that is impossible since $Ax^m + By^n = Cz^k$. So one of $\left(\frac{z}{x}\right)^2, \left(\frac{z}{y}\right)^2$ must be greater than 1 which means

$$z > y$$

when $x > y$.

Theorem 1.8. In equation (1-1) $x, y, z \geq 5$.

Proof: When $x, y, z = 2,3,4$ we have

$\{x, y, z\} = \{2,3,4\}, \{2,4,3\}, ,\{3,2,4\} ,\{3,4,2\} ,\{4,2,3\} , \{4,3,2\}$ that are all not positive integer solutions for equation (1-1) since $x, y, z > 1$ and from **Theorem 1.2** we know $x \neq y \neq z$, and the sum or difference of two even numbers can not be an odd number, and also from **Theorem 1.1** we know $\gcd(Ax, By) = \gcd(Ax, Cz) = \gcd(By, Cz) = 1$, so we have the conclusion of $x, y, z \geq 5$.

2. Proving Method

In equation (1-1), let

$$\begin{cases} a = x^{m-2} \\ b = y^{n-2} \\ c = z^{k-2} \end{cases},$$

we have

$$\begin{cases} Aax^2 + Bby^2 = Ccz^2 \\ Aa^{\frac{m-1}{m-2}}x + Bb^{\frac{n-1}{n-2}}y = Cc^{\frac{k-1}{k-2}}z \end{cases}. \tag{2-1}$$

Since we here reduce the orders of equation so the method is called “Order reducing method for equations”.

From **Theorem 1.2** we have already known that $x \neq y, x \neq z, y \neq z$, so let $x > y$ and there are four cases that need to be considered.

Case 1: $\begin{cases} y = x - f \\ z = x + e \\ Ax^{m-i} + By^{n-i} > Cz^{k-i} \end{cases};$ (2-2)

Case 2: $\begin{cases} y = x - f \\ z = x + e \\ Ax^{m-i} + By^{n-i} < Cz^{k-i} \end{cases}$; (2-3)

Case 3: $\begin{cases} y = x - f \\ z = x - e \\ Ax^{m-i} + By^{n-i} > Cz^{k-i} \end{cases}$; (2-4)

Case 4: $\begin{cases} y = x - f \\ z = x - e \\ Ax^{m-i} + By^{n-i} < Cz^{k-i} \end{cases}$. (2-5)

2.1. Case 1

From (2-1) and (2-2) we have

$$\begin{cases} Aax^2 + Bb(x - f)^2 = Cc(x + e)^2 \\ Aa^{\frac{m-1}{m-2}}x + Bb^{\frac{n-1}{n-2}}(x - f) = Cc^{\frac{k-1}{k-2}}(x + e) \end{cases}$$

and

$$\begin{cases} (Aa + Bb - Cc)x^2 - 2(Bbf + Cce)x + (Bbf^2 - Cce^2) = 0 \\ Aa^{\frac{m-1}{m-2}}x + Bb^{\frac{n-1}{n-2}}(x - f) - Cc^{\frac{k-1}{k-2}}(x + e) = 0 \end{cases},$$

the roots are

$$x = \frac{(Bbf + Cce) \pm \sqrt{(Bbf + Cce)^2 - (Aa + Bb - Cc)(Bbf^2 - Cce^2)}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} \quad (2\text{-}6)$$

and

$$x = \frac{Cc^{\frac{k-1}{k-2}}e + Bb^{\frac{n-1}{n-2}}f}{Aa^{\frac{m-1}{m-2}} + Bb^{\frac{n-1}{n-2}} - Cc^{\frac{k-1}{k-2}}} = \frac{Bbfy + Ccez}{Ax^{m-1} + By^{n-1} - Cz^{k-1}}. \quad (2\text{-}7)$$

There are two cases for Bbf^2, Cce^2 when $Bbf^2 \geq Cce^2$ and $Bbf^2 < Cce^2$.

Case A: If $Bbf^2 \geq Cce^2$, from (2-6) when

$$x = \frac{(Bbf + Cce) + \sqrt{(Bbf + Cce)^2 - (Aa + Bb - Cc)(Bbf^2 - Cce^2)}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}},$$

since from (2-2) we know $(Aa + Bb - Cc = Ax^{m-2} + By^{n-2} - Cz^{k-2}) > 0$, so we have

$$x \le \frac{2(Bbf + Cce)}{Ax^{m-2} + By^{n-2} - Cz^{k-2}},$$

and also from (2-2) since $\left(Ax^{m-1} + By^{n-1} - Cz^{n-1}\right) > 0$, compare to (2-7) we get

$$\frac{Bbfy + Ccez}{Ax^{m-1} + By^{n-1} - Cz^{k-1}} \le \frac{2(Bbf + Cce)}{Ax^{m-2} + By^{n-2} - Cz^{k-2}}.$$

From **Theorem 1.6** we know $\frac{Ax^{m-1} + By^{n-1} - Cz^{k-1}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} \le 1$, so we have

$$Bbfy + Ccez \le 2(Bbf + Cce)$$

which is impossible since from **Theorem 1.8** we know $y > 2, z > 3$.

When

$$x = \frac{(Bbf + Cce) - \sqrt{(Bbf + Cce)^2 - (Aa + Bb - Cc)(Bbf^2 - Cce^2)}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}}.$$

we have

$$x \le \frac{Bbf + Cce}{Ax^{m-2} + By^{n-2} - Cz^{k-2}},$$

compare to (2-7) we get

$$\frac{Bbfy + Ccez}{Ax^{m-1} + By^{n-1} - Cz^{k-1}} \le \frac{Bbf + Cce}{Ax^{m-2} + By^{n-2} - Cz^{k-2}}.$$

From **Theorem 1.7** we have

$$Bbfy + Ccez \le Bbf + Cce$$

which is impossible since from **Theorem 1.8** we know $y > 2, z > 3$.

Case B: If $Bbf^2 < Cce^2$, from (2-6) we have

$$(Aa + Bb - Cc)x^2 - 2(Bbf + Cce)x + (Bbf^2 - Cce^2) = 0,$$

and

$$\begin{cases} x = \dfrac{Cce^2 - Bbf^2}{(Aa + Bb - Cc)x - 2(Bbf + Cce)} \\ x > \dfrac{2(Bbf + Cce)}{(Aa + Bb - Cc)} \end{cases},$$

from (2-7) we have

$$x=\frac{Bbfy+Ccez}{Ax^{m-1}+By^{n-1}-Cz^{k-1}}=\frac{Bbf(x-f)+Cce(x+e)}{Ax^{m-1}+By^{n-1}-Cz^{k-1}},$$

in which

$$\begin{cases} x=\dfrac{Cce^2-Bbf^2}{\left(Ax^{m-1}+By^{n-1}-Cz^{k-1}\right)-(Bbf+Cce)}, \\ Ax^{m-1}+By^{n-1}-Cz^{k-1}>Bbf+Cce \end{cases}$$

from **Theorem 1.6** we have

$$\left(Aa+Bb-Cc=Ax^{m-2}+By^{n-2}-Cz^{k-2}\right)\geq\left(Ax^{m-1}+By^{n-1}-Cz^{k-1}\right)>(Bbf+Cce),$$

so we get

$$x=\frac{Cce^2-Bbf^2}{(Aa+Bb-Cc)x-2(Bbf+Cce)}=\frac{Cce^2-Bbf^2}{\left(Ax^{m-1}+By^{n-1}-Cz^{k-1}\right)-(Bbf+Cce)}$$

where

$$(Aa+bB-cC)x-2(Bbf+Cce)=\left(Ax^{m-1}+By^{n-1}-Cz^{k-1}\right)-(Bbf+Cce),$$

and

$$x=\frac{\left(Ax^{m-1}+By^{n-1}-Cz^{k-1}\right)+(Bbf+Cce)}{(Aa+Bb-Cc)}<2,$$

that is impossible since $x>1$.

From (2-6) when

$$x=\frac{(Bbf+Cce)-\sqrt{(Bbf+Cce)^2+(aA+Bb-cC)\left(Cce^2-Bbf^2\right)}}{Ax^{m-2}+By^{n-2}-Cz^{k-2}}$$

is not possible since $x\leq 0$.

2.2. Case 2

From (2-1) and (2-3) we have

$$\begin{cases} Aax^2+Bb(x-f)^2=Cc(x+e)^2 \\ Aa^{\frac{m-1}{m-2}}x+Bb^{\frac{n-1}{n-2}}(x-f)=Cc^{\frac{k-1}{k-2}}(x+e) \end{cases}$$

and

$$\begin{cases} (Aa+Bb-Cc)x^2-2(Bbf+Cce)x+\left(Bbf^2-Cce^2\right)=0 \\ Aa^{\frac{m-1}{m-2}}x+Bb^{\frac{n-1}{n-2}}(x-f)-Cc^{\frac{k-1}{k-2}}(x+e)=0 \end{cases},$$

the roots are

$$x = \frac{(Bbf + Cce) \pm \sqrt{(Bbf + Cce)^2 - (Aa + Bb - Cc)(Bbf^2 - Cce^2)}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}}$$

and

$$x = \frac{Cc^{\frac{k-1}{k-2}} e + Bb^{\frac{n-1}{n-2}} f}{Aa^{\frac{m-1}{m-2}} + Bb^{\frac{n-1}{n-2}} - Cc^{\frac{k-1}{k-2}}} = \frac{Bbfy + Ccez}{Ax^{m-1} + By^{n-1} - Cz^{k-1}} .$$

Since from (2-3) we know $\left(Ax^{m-1} + By^{n-1} - Cz^{k-1}\right) < 0$, so we have $x < 0$ that is impossible.

2.3. Case 3

From (2-1) and (2-4) we have

$$\begin{cases} Aax^2 + Bb(x - f)^2 = Cc(x - e)^2 \\ Aa^{\frac{m-1}{m-2}} x + Bb^{\frac{n-1}{n-2}} (x - f) = Cc^{\frac{k-1}{k-2}} (x - e) \end{cases}$$

and

$$\begin{cases} (Aa + Bb - Cc)x^2 - 2(Bbf - Cce)x + \left(Bbf^2 - Cce^2\right) = 0 \\ Aa^{\frac{m-1}{m-2}} x + Bb^{\frac{n-1}{n-2}} (x - f) - Cc^{\frac{k-1}{k-2}} (x - e) = 0 \end{cases} ,$$

the roots are

$$x = \frac{(Bbf - Cce) \pm \sqrt{(Bbf - Cce)^2 - (Aa + Bb - Cc)(Bbf^2 - Cce^2)}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} \qquad (2\text{-}8)$$

and

$$x = \frac{Bb^{\frac{n-1}{n-2}} f - Cc^{\frac{k-1}{k-2}} e}{Aa^{\frac{m-1}{m-2}} + Bb^{\frac{n-1}{n-2}} - Cc^{\frac{k-1}{k-2}}} = \frac{Bbfy - Ccez}{Ax^{m-1} + By^{n-1} - Cz^{k-1}} . \qquad (2\text{-}9)$$

There are two cases for Bbf^2, Cce^2 when $Bbf^2 \geq Cce^2$ and $Bbf^2 < Cce^2$.

Case A: If $Bbf^2 \geq Cce^2$, from (2-8) when

$$x = \frac{(Bbf - Cce) + \sqrt{(Bbf - Cce)^2 - (Aa + Bb - Cc)(Bbf^2 - Cce^2)}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} ,$$

then $Bbf \leq Cce$ is not possible since causes $x \leq 0$ or negative value under the root, so we have $Bbf > Cce$, and from (2-4) since $\left(Ax^{m-1} + By^{n-1} - Cz^{k-1}\right) > 0$, compare to (2-9) we get

$$\begin{cases} Bbf > Cce \\ \dfrac{Bbfy - Ccez}{Ax^{m-1} + By^{n-1} - Cz^{k-1}} = \dfrac{(Bbf - Cce) + \sqrt{(Bbf - Cce)^2 - (Aa + Bb - Cc)(Bbf^2 - Cce^2)}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} \\ Bbfy - Ccez > 0 \end{cases}.$$

From **Theorem 1.6** we know $\dfrac{Ax^{m-1} + By^{n-1} - Cz^{k-1}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} \le 1$, so we have

$$0 < Bbfy - Ccez \le (Bbf - Cce) + \sqrt{(Bbf - Cce)^2 - (Aa + Bb - Cc)(Bbf^2 - Cce^2)},$$

and

$$0 < Bbfy - Ccez \le 2(Bbf - Cce) \Rightarrow \left(\frac{z}{y} < \frac{Bbf}{Cce} \le \frac{z-2}{y-2}\right) \Rightarrow (z > y, e < f)$$

where

$$\begin{cases} Bbf(x - f) - Cce(x - e) \le 2(Cbf - Cce) \\ Bbf(x - f) - Cce(x - e) > 0 \end{cases}$$

and

$$\left(\frac{Bbf^2 - Cce^2}{Bbf - Cce}\right) < x \le \left[\left(2 + \frac{Bbf^2 - Cce^2}{Bbf - Cce}\right) = \left(2 + \frac{\left(\frac{Bbf}{Cce} - \frac{e}{f}\right)f}{\frac{Bbf}{Cce} - 1}\right)\right] < \left(2 + \frac{\left(\frac{Bbf}{Cce}\right)f}{\frac{Bbf}{Cce} - 1}\right),$$

from **Theorem 1.8** we know $x, y, z \ge 5$ which means $x > 3$ and

$$\frac{\left(\frac{Bbf}{Cce}\right)f}{\frac{Bbf}{Cce} - 1} > 1,$$

in which

$$0 < 1 < \left(\frac{Bbf}{Cce}\right) < \frac{1}{1 - f},$$

so we get

$$0 < f < 1,$$

that is impossible.

When

$$x=\frac{(Bbf-Cce)-\sqrt{(Bbf-Cce)^2-(Aa+Bb-Cc)(Bbf^2-Cce^2)}}{Ax^{m-2}+By^{n-2}-Cz^{k-2}},$$

if $Bbf\le Cce$ then $x\le 0$ which is not possible, so we have

$$\begin{cases} x\le\dfrac{(Bbf-Cce)}{Ax^{m-2}+By^{n-2}-Cz^{k-2}}, \\ Bbf>Cce \end{cases}$$

compare to (2-9) we get

$$\begin{cases} \dfrac{Bbfy-Ccez}{Ax^{m-1}+By^{n-1}-Cz^{k-1}}\le\dfrac{(Bbf-Cce)}{x^{m-2}+By^{n-2}-Cz^{k-2}}. \\ Bbfy-Ccez>0 \end{cases}$$

From **Theorem 1.6** we have

$$0<Bbfy-Ccez\le(Bbf-Cce)\Rightarrow\left(\frac{z}{y}<\frac{Bbf}{Cce}\le\frac{z-1}{y-1}\right)\Rightarrow(z>y,e<f)$$

and

$$0<Bbf(x-f)-Cce(x-e)\le(Bbf-Cce),$$

where

$$\left(\frac{Bbf^2-Cce^2}{Bbf-Cce}\right)<x\le\left[\left(1+\frac{Bbf^2-Cce^2}{Bbf-Cce}\right)=\left(1+\frac{\left(\frac{Bbf}{Cce}-\frac{e}{f}\right)f}{\frac{Bbf}{Cce}-1}\right)\right]<\left(1+\frac{\left(\frac{Bbf}{Cce}\right)f}{\frac{Bbf}{Cce}-1}\right),$$

from **Theorem 1.8** we know $x,y,z\ge 5$ which means $x>3$ and

$$\frac{\left(\frac{Bbf}{Cce}\right)f}{\frac{Bbf}{Cce}-1}>1,$$

in which

$$0<1<\left(\frac{Bbf}{Cce}\right)<\frac{1}{1-f},$$

so we get

$$0<f<1,$$

that is impossible.

Case B: If $Bbf^2<Cce^2$, from (2-8) we have

$$x=\frac{(Bbf-Cce)+\sqrt{(Bbf-Cce)^2+(Aa+Bb-Cc)(Cce^2-Bbf^2)}}{Ax^{m-2}+By^{n-2}-Cz^{k-2}},$$

compare to (2-9) we get

$$\frac{Bbfy-Ccez}{Ax^{m-1}+By^{n-1}-Cz^{k-1}}=\frac{(Bbf-Cce)+\sqrt{(Bbf-Cce)^2+(Aa+Bb-Cc)(Cce^2-Bbf^2)}}{Ax^{m-2}+By^{n-2}-Cz^{k-2}}.$$

From **Theorem 1.6**, when $Bbf>Cce$, we have

$$\begin{cases}0<Bbfy-Ccez\le(Bbf-Cce)\left(1+\sqrt{1+r}\right)\\ \left[\frac{z}{y}<\frac{Bbf}{Cce}<\frac{z-\left(1+\sqrt{1+r}\right)}{y-\left(1+\sqrt{1+r}\right)}\right]\Rightarrow(z>y),(f>e)\\ (Aa+Bb-Cc)(Cce^2-Bbf^2)=r(Bbf-Cce)^2\\ \frac{Bbf^2-Cce^2}{Bbf-Cce}<x\le\left(1+\sqrt{1+r}+\frac{Bbf^2-Cce^2}{Bbf-Cce}\right)<1+\sqrt{1+r}\end{cases},$$

but since $Bbf^2<Cce^2$ so we have

$$\frac{Bbf}{Cce}<\frac{e}{f}<1$$

that is impossible since contradicts against $Bbf>Cce$. If $r=0$ then we get $Cce^2-Bbf^2=0$ and $x<2$ that is also impossible.

When $Bbf=Cce$, we have

$$\begin{cases}Bbfy-Ccez\le\sqrt{(Aa+Bb-Cc)(Cce^2-Bbf^2)}\Rightarrow Bbf(e-f)\le(Aa+Bb-Cc)\Rightarrow\\ \Rightarrow(Aa+Bb-Cc)\ge(Cce^2-Bbf^2)\Rightarrow(x\le1)\end{cases}$$

that is impossible.

When $Bbf<Cce$, we have

$$\begin{cases} 0 < Bbfy - Ccez \le (Cce - Bbf)\left(-1 + \sqrt{1+r}\right) \\ \left[\dfrac{z}{y} < \dfrac{Bbf}{Cce} < \dfrac{z + \left(-1+\sqrt{1+r}\right)}{y + \left(-1+\sqrt{1+r}\right)}\right] \Rightarrow (y > z), (e > f) \\ (Aa + Bb - Cc)\left(Cce^2 - Bbf^2\right) = r(Cce - Bbf)^2 \\ \left(\dfrac{Bbf^2 - Cce^2}{Bbf - Cce}\right) > x \ge \left(1 - \sqrt{1+r} + \dfrac{Bbf^2 - Cce^2}{Bbf - Cce}\right) \end{cases},$$

that is impossible since from **Theorem 1.7** we know $y < z$. If $r = 0$ then we get $Cce^2 - Bbf^2 = 0$ and $x < 0$ that is also impossible.

From (2-6) when

$$x = \frac{(Bbf - Cce) - \sqrt{(Bbf - Cce)^2 + (Aa + Bb - Cc)\left(Cce^2 - Bbf^2\right)}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}}$$

is not possible since $x \le 0$.

2.4. Case 4

From (2-1) and (2-5) we have

$$\begin{cases} Aax^2 + Bb(x - f)^2 = Cc(x - e)^2 \\ Aa^{\frac{m-1}{m-2}}x + Bb^{\frac{n-1}{n-2}}(x - f) = Cc^{\frac{k-1}{k-2}}(x - e) \end{cases}$$

and

$$\begin{cases} (Aa + Bb - Cc)x^2 - 2(Bbf - Cce)x + \left(Bbf^2 - Cce^2\right) = 0 \\ Aa^{\frac{m-1}{m-2}}x + Bb^{\frac{n-1}{n-2}}(x - f) - Cc^{\frac{k-1}{k-2}}(x - e) = 0 \end{cases},$$

the roots are

$$x = \frac{(Bbf - Cce) \pm \sqrt{(Bbf - Cce)^2 + (Cc - Aa - Bb)\left(Bbf^2 - Cce^2\right)}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} \tag{2-13}$$

and

$$x = \frac{Bb^{\frac{n-1}{n-2}} f - Cc^{\frac{k-1}{k-2}} e}{Aa^{\frac{m-1}{m-2}} + Bb^{\frac{n-1}{n-2}} - Cc^{\frac{k-1}{k-2}}} = \frac{Bbfy - Ccez}{Ax^{m-1} + By^{n-1} - Cz^{k-1}}. \tag{2-14}$$

There are two cases for Bbf^2, Cce^2 when $Bbf^2 \ge Cce^2$ and $Bbf^2 < Cce^2$.

Case A: If $Bbf^2 \ge Cce^2$, from (2-13) when

$$x=\frac{(Bbf-Cce)+\sqrt{(Bbf-Cce)^2+(Cc-Aa-Bb)(Bbf^2-Cce^2)}}{Ax^{m-2}+By^{n-2}-Cz^{k-2}}$$

is impossible since $x\le 0$.

When

$$x=\frac{(Bbf-Cce)-\sqrt{(Bbf-Cce)^2+(Cc-Aa-Bb)(Bbf^2-Cce^2)}}{Ax^{m-2}+By^{n-2}-Cz^{k-2}},$$

compare to (2-14) we get

$$\frac{Bbfy-Ccez}{Ax^{m-1}+By^{n-1}-Cz^{k-1}}=\frac{(Bbf-Cce)-\sqrt{(Bbf-Cce)^2+(Cc-Aa-Bb)(Bbf^2-Cce^2)}}{Ax^{m-2}+By^{n-2}-Cz^{k-2}}.$$

From **Theorem 1.5** we know $\frac{Ax^{m-1}+By^{n-1}-Cz^{k-1}}{Ax^{m-2}+By^{n-2}-Cz^{k-2}}<\frac{1}{2}$, so when $Bbf>Cce$, we have

$$\begin{cases} 0<Ccez-Bbfy<\frac{-1+\sqrt{1+r}}{2}(Bbf-Cce) \\ (Cc-Aa-Bb)(Bbf^2-Cce^2)=r(Bbf-Cce)^2 \\ \left[\frac{Bbf^2-Cce^2}{Bbf-Cce}=\frac{r(Bbf-Cce)}{Cc-Aa-Bb}\right]\Rightarrow\left(\frac{r(Bbf-Cce)}{Cc-Aa-Bb}=\frac{rx}{\frac{-1+\sqrt{1+r}}{2}}\right) \\ \frac{\left(\frac{Bbf}{Cce}\right)f}{\frac{Bbf}{Cce}-1}>\left(\frac{Bbf^2-Cce^2}{Bbf-Cce}\right)>x>\left(\frac{1-\sqrt{1+r}}{2}+\frac{Bbf^2-Cce^2}{Bbf-Cce}\right) \end{cases},$$

we get

$$x>\frac{1-\sqrt{1+r}}{2}+\frac{rx}{\frac{-1+\sqrt{1+r}}{2}}$$

which means

$$\frac{\left(\frac{-1+\sqrt{1+r}}{2}\right)^2}{r-\frac{-1+\sqrt{1+r}}{2}}>x,$$

since $x>1$, so we have

$$\left(\frac{-1+\sqrt{1+r}}{2}\right)^2+\frac{-1+\sqrt{1+r}}{2}>r$$

and

$$r > 4r$$

that is impossible. If $r = 0$ then we get $Cce^2 - Bbf^2 = 0$ and $x < 0$ that is also impossible.

When $Bbf = Cce$, we have

$$\begin{cases} Ccez - Bbfy < \sqrt{(Cc - Aa - Bb)(Bbf^2 - Cce^2)} \Rightarrow Bbf(f - e) < (Cc - Aa - Bb) \Rightarrow \\ \Rightarrow (Cc - Aa - Bb) > (Bbf^2 - Cce^2) \Rightarrow (x < 1) \end{cases}$$

that is impossible.

When $Bbf < Cce$, we have

$$\begin{cases} 0 < Ccez - Bbfy < \dfrac{1 + \sqrt{1+r}}{2}(Cce - Bbf) \\ (Cc - Aa - Bb)(Bbf^2 - Cce^2) = r(Cce - Bbf)^2 \\ \left[\dfrac{Cce^2 - Bbf^2}{Cce - Bbf} = \dfrac{r(Cce - Bbf)}{Cc - Aa - Bb}\right] < x < \left(\dfrac{1+\sqrt{1+r}}{2} + \dfrac{Cce^2 - Bbf^2}{Cce - Bbf}\right) < \left(\dfrac{1+\sqrt{1+r}}{2}\right) \end{cases}$$

that is impossible since $\dfrac{Cce^2 - Bbf^2}{Cce - Bbf} < 0$ but $\dfrac{r(Cce - Bbf)}{Cc - Aa - Bb} > 0$, so

$$\frac{Cce^2 - Bbf^2}{Cce - Bbf} \neq \frac{r(Cce - Bbf)}{Cc - Aa - Bb}.$$

If $r = 0$ then we get $Cce^2 - Bbf^2 = 0$ and

$$x < \left[\left(\frac{1+\sqrt{1+r}}{2}\right) = 1\right]$$

that is also impossible.

Case B: If $Bbf^2 < Cce^2$, from (2-13) when

$$x = \frac{(Bbf - Cce) + \sqrt{(Bbf - Cce)^2 - (Cc - Aa - Bb)(Cce^2 - Bbf^2)}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}},$$

obviously it is impossible when $Bbf \geq Cce$ since $x \leq 0$ or negative value under the root. So we have $Bbf < Cce$, compare to (2-14) we get

$$\begin{cases} \dfrac{Bbfy - Ccez}{Ax^{m-1} + By^{n-1} - Cz^{k-1}} = \dfrac{(Bbf - Cce) + \sqrt{(Bbf - Cce)^2 - (Cc - Aa - Bb)(Cce^2 - Bbf^2)}}{Ax^{m-2} + By^{n-2} - Cz^{k-2}} \\ Bbfy - Ccez < 0 \end{cases}.$$

From **Theorem 1.5** we have

$$0 < Ccez - Bbfy < \frac{(Cce - Bbf) - \sqrt{(Cce - Bbf)^2 - (Cc - Aa - Bb)(Cce^2 - Bbf^2)}}{2}$$

where

$$\begin{cases} Bbf < Cce \\ 0 < Cce(x-e) - Bbf(x-f) < \dfrac{(Cce - Bbf)\left(1 - \sqrt{1-r}\right)}{2} \\ \left(\dfrac{y}{z} < \dfrac{Cce}{Bbf} < \dfrac{y-0.5}{z-0.5}\right) \Rightarrow (y > z), (e > f) \\ \left[(Cc - Aa - Bb)(Cce^2 - Bbf^2) = r(Cce - Bbf)^2\right] \Rightarrow \\ \left[\dfrac{Cce^2 - Bbf^2}{Cce - Bbf} = \dfrac{r(Cce - Bbf)}{Cc - Aa - Bb} = \dfrac{rx}{\frac{1-\sqrt{1-r}}{2}}\right] \\ \dfrac{ce^2 - bf^2}{ce - bf} < x < \left(\dfrac{1-\sqrt{1-r}}{2} + \dfrac{ce^2 - bf^2}{ce - bf}\right) \end{cases},$$

we get

$$x < \frac{1-\sqrt{1-r}}{2} + \frac{rx}{\frac{1-\sqrt{1-r}}{2}}$$

which means

$$x < \frac{\left(\frac{1-\sqrt{1-r}}{2}\right)^2}{\frac{1-\sqrt{1-r}}{2} + r}$$

since $x > 1$, so we have

$$4r < -r$$

that is impossible. If $r = 0$ then we get $Cce^2 - Bbf^2 = 0$ and

$$x < \left[\left(\frac{1-\sqrt{1-r}}{2} + \frac{Cce^2 - Bbf^2}{Cce - Bbf}\right) = 0\right]$$

that is also impossible.

When

$$x=\frac{(Bbf-Cce)-\sqrt{(Bbf-Cce)^2-(Cc-Aa-Bb)(Cce^2-Bbf^2)}}{Ax^{m-2}+By^{n-2}-Cz^{k-2}},$$

obviously it is impossible when $Bbf \geq Cce$ since $x \leq 0$ or negative value under the root. So we have $Bbf < Cce$, compare to (2-14) we get

$$\begin{cases}\dfrac{Bbfy-Ccez}{x^{m-1}+y^{n-1}-z^{k-1}}=\dfrac{(Bbf-Cce)-\sqrt{(Bbf-Cce)^2-(Cc-Aa-Bb)(Cce^2-Bbf^2)}}{Ax^{m-2}+By^{n-2}-Cz^{k-2}}\\ Bbfy-Ccez<0\end{cases}.$$

From **Theorem 1.5** we have

$$0<Ccez-Bbfy<\frac{(Cce-Bbf)+\sqrt{(Cce-Bbf)^2-(Cc-Aa-Bb)(Cce^2-Bbf^2)}}{2}$$

where

$$\begin{cases}Bbf<Cce\\ 0<Cce(x-e)-Bbf(x-f)<\dfrac{(Cce-Bbf)\left(1+\sqrt{1-r}\right)}{2}\\ \left[(Cc-aA-Bb)(Cce^2-Bbf^2)=r(Cce-Bbf)^2\right]\Rightarrow\\ \left[\dfrac{Cce^2-Bbf^2}{Cce-Bbf}=\dfrac{r(Cce-Bbf)}{Cc-Aa-Bb}=\dfrac{rx}{\frac{1+\sqrt{1-r}}{2}}\right]\\ \dfrac{Cce^2-Bbf^2}{Cce-Bbf}<x<\left(\dfrac{1+\sqrt{1-r}}{2}+\dfrac{Cce^2-Bbf^2}{Cce-Bbf}\right)\end{cases},$$

we get

$$x<\frac{1+\sqrt{1-r}}{2}+\frac{rx}{\frac{1+\sqrt{1-r}}{2}}$$

which means

$$x<\frac{\left(\frac{1+\sqrt{1-r}}{2}\right)^2}{\frac{1+\sqrt{1-r}}{2}+r}$$

since $x>1$, so we have

$$4r<-r$$

that is impossible. If $r=0$ then we get $Cce^2 - Bbf^2 = 0$ and

$$x < \left[\left(\frac{1+\sqrt{1-r}}{2} + \frac{Cce^2 - Bbf^2}{Cce - Bbf}\right) = 1\right]$$

that is also impossible.

3. Conclusion

Through the above contents we can see clearly that "Order reducing method for equations" is perfect to prove equation (1-1) to have no positive integer solutions, and the great benefit from "Order reducing method for equations" is that there is no need to consider the exponents of m, n, k and all the numbers for them can be proved together.