On gravitational quantum wavefunctions

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Abstract

A gravitational momentum space wavefunction and its inverse spacetime Fourier transformation are constructed from the classical particle Lagrangian. From non-relativistic metric, the momentum space wavefunction gives the Maxwell-Boltzmann distribution with gravity temperature relation and the spacetime wavefunction gives the Schrodinger equation with a harmonic oscillator potential, leading to the Gauss's law for gravity with mass density gravity relation. From the special relativity metric, which is shown to set the temperature at about $7.64 \times 10^{-12} K$ and density at about $3.58 \times 10^{9} kgm^{-3}$, the momentum space wavefunction gives a relativistic momentum distribution and the spacetime wavefunction gives the Dirac equation with a spacetime potential. Finally, the many identical particles wavefunctions are also constructed.

I. Introduction

In this work, the momentum space wavefunction and its inverse spacetime Fourier transformation is constructed from the classical particle Lagrangian L of a particle with mass m and momentum vector $\boldsymbol{p}$ in a gravitational field with metric matrix $\boldsymbol{g}$, which can be expressed in the form

$$L = -\frac{1}{2} tr\left[\frac{p_\mu g_{\mu\nu}^{-1} p_\nu}{m}\right] \tag{1}$$

From non-relativistic metric, the momentum space wavefunction gives the Maxwell-Boltzmann distribution with temperature gravity relation and the spacetime wavefunction gives the Schrodinger equation with a harmonic oscillator potential, which leads to the Gauss's law for gravity with mass density gravity relation. From special relativity metric, which is shown to set the temperature at about $7.64 \times 10^{-12} K$ and density at about $3.58 \times 10^{9} kgm^{-3}$, the momentum space wavefunction gives a relativistic momentum distribution and the spacetime wavefunction gives the Dirac equation with a spacetime potential. Finally, the many identical particles wavefunctions are also constructed.

II. Wavefunctions

a. Momentum space wavefunction

From the Lagrangian in equation (1), let us construct a normalized [1] momentum space wavefunction $\psi(\boldsymbol{p})$ of the particle with momentum vector $\boldsymbol{p}$ in a gravitational field with metric matrix $\boldsymbol{g}$ as

$$\psi(\boldsymbol{p}) = \det\left[\frac{p_\mu g_{\mu\nu}^{-1} p_\nu}{hm}\right]^{\frac{1}{2}} \exp\left[-\frac{1}{2} tr\left[\frac{p_\mu g_{\mu\nu}^{-1} p_\nu}{\hbar m}\right]\right] \tag{2}$$

b. Momentum space wavefunction Fourier transformation

The spacetime wavefunction $\psi(\boldsymbol{x})$ is given by the Fourier transformation [2] of the momentum space wavefunction $\psi(\boldsymbol{p})$ in equation (2) as

$$\psi(\boldsymbol{x}) = \int_{-\infty}^{\infty} \frac{d^n P}{\hbar^n} \det\left[\frac{p_\mu g_{\mu\nu}^{-1} p_\nu}{hm}\right]^{-\frac{1}{2}} \psi(\boldsymbol{p})\psi(\boldsymbol{p},\boldsymbol{x}) \tag{3}$$

With

$$\psi(\boldsymbol{p},\boldsymbol{x}) = \exp\left[-\frac{i}{\hbar} tr\left[p_\mu x_\nu\right]\right] \tag{4}$$

Using equations (2) and (4) in equation (3) gives the spacetime wavefunction

$$\psi(\boldsymbol{x}) = det\left[\frac{m x_\mu g_{\mu\nu} x_\nu}{h}\right]^{\frac{1}{2}} \exp\left[-\frac{1}{2} tr\left[\frac{m x_\mu g_{\mu\nu} x_\nu}{\hbar}\right]\right] \tag{5}$$

a. Spacetime wavefunction inverse Fourier transformation

The inverse Fourier transformation of the spacetime wavefunction in equation (5) is given by

$$\psi(\boldsymbol{p}) = \int_{-\infty}^{\infty} d^n x det\left[\frac{m x_\mu g_{\mu\nu} x_\nu}{h}\right]^{-\frac{1}{2}} \psi(\boldsymbol{x})\psi^*(\boldsymbol{p},\boldsymbol{x}) \tag{6}$$

Evaluating equation (6) gives back the momentum wave function in equation (2)

$$\psi(\boldsymbol{p}) = \det\left[\frac{p_\mu g_{\mu\nu}^{-1} p_\nu}{hm}\right]^{\frac{1}{2}} \exp\left[-\frac{1}{2} tr\left[\frac{p_\mu g_{\mu\nu}^{-1} p_\nu}{\hbar m}\right]\right] \tag{7}$$

III. Non-relativistic wavefunctions

In non-relativistic quantum and classical mechanics, the momentum $\boldsymbol{p}$, position $\boldsymbol{x}$ and gravity field $\boldsymbol{g}$ have the following form

$$\boldsymbol{p} = m\boldsymbol{v}, \qquad \boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \boldsymbol{g} = \begin{bmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{bmatrix} \tag{8}$$

a. Velocity space wavefunction

Substituting equations (8) in equation (7) gives the velocity wavefunction

$$\psi(\boldsymbol{v}) = v_x v_y v_z \left[\frac{m}{\hbar g}\right]^{\frac{3}{2}} \exp\left[-\frac{1}{2}\frac{m\boldsymbol{v}^2}{\hbar g}\right] \tag{9}$$

Equation (9) and the Maxwell-Boltzmann distribution [3] gives a temperature and gravity relation

$$T = \frac{\hbar g}{k} \tag{10}$$

b. Spatial wavefunction

Substituting equations (8) in equation (5) gives the spatial wavefunction of a particle as

$$\psi(\boldsymbol{x}) = xyz\left[\frac{mg}{\hbar}\right]^{\frac{3}{2}} \exp\left[-\frac{1}{2}\frac{mg\boldsymbol{x}^2}{\hbar}\right] \tag{11}$$

Equation (11) is the solution of the Schrödinger equation [4] with the harmonic oscillator potential

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}mg^2\,\boldsymbol{x}^2\right]\psi(\boldsymbol{x}) = 3\hbar g\,\psi(\boldsymbol{x}) \tag{12}$$

c. Hooke-Newton's Law

Taking the gradient of the potential in equation (12) gives the Hooke's law

$$\boldsymbol{F} = -mg^2\boldsymbol{x} \tag{13}$$

Using Newton's law $\boldsymbol{F} = m\boldsymbol{a}$ in equation (13) gives

$$\boldsymbol{a} = -g^2\boldsymbol{x} \tag{14}$$

Taking the divergence of the acceleration in equation (14) gives Gauss's law [5] for gravity

$$\boldsymbol{\nabla} \cdot \boldsymbol{a} = -3g^2 \tag{15}$$

With mass density gravity relation

$$\rho = \frac{3g^2}{4\pi G} \tag{16}$$

IV. Special relativistic wavefunctions

For special relativistic wavefunctions, the momentum $\boldsymbol{p}$, position $\boldsymbol{x}$ and gravity field $\boldsymbol{g}$ have the following form

$$\boldsymbol{p} = \begin{bmatrix} p_{ct} \\ p_x \\ p_y \\ p_z \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \text{ and } \boldsymbol{g} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

From the temperature and density metric relations in equations (10) and (16), the special relativity metric in equation (17) set the temperature at about $7.64 \times 10^{-12} K$ and density at about $3.58 \times 10^{9} kgm^{-3}$.

a. Momentum space wavefunction

Substituting equations (17) in equation (7) gives momentum wavefunction

$$\psi(\boldsymbol{p}) = \frac{i p_{ct} p_x p_y p_z}{h^2 m^2} \exp\left[-\frac{1}{2}\frac{\boldsymbol{p}^2}{\hbar m}\right] \tag{18}$$

b. Spacetime wavefunction

Substituting equations (17) in equation (6) gives the spacetime wavefunction of a particle as

$$\psi(\boldsymbol{x}) = \frac{m^2 ictxyz}{h^2} \exp\left[-\frac{1}{2}\frac{m\boldsymbol{x}^2}{\hbar}\right] \tag{19}$$

Equation (19) is the solution of the wave equation with a potential

$$\left[\Box - \frac{m^2 \boldsymbol{x}^2}{\hbar^2}\right] \psi(\boldsymbol{x}) = -\frac{8m}{\hbar}\, \psi(\boldsymbol{x}) \tag{20}$$

From equation (20), the Dirac equation [6] with spacetime potential can be written as

$$\gamma^{\mu}\left[\partial_{\mu} + i\frac{m}{\hbar} x_{\mu}\right] \psi(\boldsymbol{x}) = 0 \tag{21}$$

V. Many particles

From equations (2) and (5), the many identical particles momentum space wavefunction $\psi(\boldsymbol{p}; n)$ and the spacetime wavefunction $\psi(\boldsymbol{x}; n)$ are given by

$$\psi(\boldsymbol{p}; n) = \frac{1}{\sqrt{n!}} \det\left[\frac{p_{\mu} g_{\mu\nu}^{-1} p_{\nu}}{hm}\right]^{\frac{n}{2}} \exp\left[-\frac{1}{2} tr\left[\frac{p_{\mu} g_{\mu\nu}^{-1} p_{\nu}}{\hbar m}\right]\right] \tag{21}$$

$$\psi(\boldsymbol{x};n) = \frac{1}{\sqrt{n!}} det\left[\frac{mx_{\mu}g_{\mu\nu}x_{\nu}}{h}\right]^{\frac{n}{2}} \exp\left[-\frac{1}{2}tr\left[\frac{mx_{\mu}g_{\mu\nu}x_{\nu}}{\hbar}\right]\right] \quad (22)$$

VI. Conclusion

In summary, the momentum space wavefunction and its inverse spacetime Fourier transformation were constructed from the classical particle Lagrangian. From non-relativistic metric, the momentum space wavefunction gave the Maxwell-Boltzmann distribution with temperature gravity relation and the spacetime wavefunction gave the Schrodinger equation with a harmonic oscillator potential, which lead to the Gauss's law for gravity with mass density gravity relation. From special relativity metric, which is shown to set the temperature at about $7.64 \times 10^{-12} K$ and density at about $3.58 \times 10^{9} kgm^{-3}$, the momentum space wavefunction gave a relativistic momentum distribution and the spacetime wavefunction gave the Dirac equation with spacetime a potential. Finally, the many identical particles wavefunctions are also constructed.

VII. References

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