GENERAL RELATIVISTIC FORMULATION OF QUANTUM MECHANICS

Vu B Ho

Advance Study, 9 Adela Court, Mulgrave, Victoria 3170, Australia

Email: vubho@bigpond.net.au

Abstract: In this work we show that it is possible to formulate quantum mechanics from general relativity in both pseudo-Euclidean and Euclidean metric by showing that the three-dimensional differentiable spacetime structure of a quantum particle can be converted to that of a manifestly Minkowski spacetime or a manifestly Euclidean spacetime. This is equivalent to viewing and describing three-dimensional quantum particles as normal particles in classical and quantum mechanics.

In our previous works on the quantum structures of elementary particles, we considered elementary particles as three-dimensional differentiable manifolds and accordingly described their dynamics as the dynamics of three-dimensional differentiable manifolds in an ambient Euclidean space. Furthermore, if elementary particles are assumed to remain as stable structures then their intrinsic dynamics should be described by continuous isometric embeddings [1]. Even though such a fundamental dynamics formulated in terms of differential geometry and topology is essential for an attempt to reconcile general relativity and quantum mechanics, a question that arises is whether it is possible to investigate their dynamics using the normal experimental procedure. In experimental physics, in order to investigate the dynamics of elementary particles we need to view them as physical objects in a Minkowski spacetime. In our present situation, this view requires to transform the three-dimensional spacetime structure of a quantum particle to that of a manifestly Minkowski spacetime and the spacetime structure of this manifestly Minkowski manifold in turns can be represented by physical quantities that represent the particle, such as wavefunctions in quantum mechanics. It is often stated that general relativity may not be compatible with quantum theory because the former is formulated in terms of curved spacetimes while the latter is formulated from the view of an observer in flat Minkowski spacetime in which the quantum dynamics of a particle is described in terms of a Hilbert space of physical states. In our previous works on the spacetime structures of quantum particles we showed that general relativity can be reconciled with quantum mechanics by formulating general relativity purely in terms of differential geometry by considering Bianchi identities as fundamental field equations for the gravitational field instead of Einstein field equations. By this way, Schrödinger wavefunctions are employed as mathematical objects to construct the spacetime of a quantum particle [2]. However, in this work we will approach the problem in a more intuitive manner and show that general relativity and quantum mechanics can be reconciled if the curved spacetime of a quantum particle is constructed in such manner that it can be transformed to a manifestly Minkowski spacetime and the dynamics of a particle can be deduced from the mathematical formulation of general relativity. We will discuss for two different cases in which quantum mechanics can be formulated from either the pseudo-Euclidean general relativity or the Euclidean general relativity. Instead of formulating a complete description of the motion of quantum particles as isometric embedding in an ambient Euclidean space we show that it is possible to convert their three-dimensional differentiable structures to that of a manifestly Minkowski space so that even though the true spacetime structure of a quantum particle is a three-dimensional differentiable manifold it can be described as a particle in a Minkowksi space.

1. Pseudo-Euclidean General Relativistic Formulation of Quantum Mechanics

In classical physics, the Minkowski spacetime was formulated to provide a mathematical background for Einstein theory of special relativity. In special relativity, the Lorentz coordinate transformation between the inertial frame S with spacetime coordinates (ct, x, y, z) and the inertial frame S' with coordinates (ct', x', y', z') is derived from the principle of relativity and the postulate of a universal speed c

$$x' = \gamma(x - \beta ct) \tag{1}$$

$$y' = y \tag{2}$$

$$z' = z \tag{3}$$

$$ct' = \gamma(-\beta x + ct) \tag{4}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$ [3]. It can be verified that the Lorentz transformation given in Equations (1-4) leaves the Minkowski spacetime interval $-c^2t^2 + x^2 + y^2 + z^2$ invariant. Spacetime with this metric is a pseudo-Euclidean space. Even though the universal speed in the Lorentz transformation was assumed to be the speed of light in vacuum, with the assumption of the pseudo-Euclidean metric, Einstein then generalised special relativity to general relativity to describe the gravitational field in which the speed of the gravitational interaction was also assumed to take the value of the universal speed. Why the speed of the gravitational interacton and the speed of an electromagnetic wave, which are assumed to have different physical backgrounds, should be the same is a profound question that is needed to be investigated. The Einstein general relativistic field equations of the gravitational field are proposed to take the form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{5}$$

where Λ and κ are undetermined constants, $T_{\alpha\beta}$ is the energy-momentum tensor and $R_{\alpha\beta}$ is the Ricci tensor defined in terms of the affine connection $\Gamma^{\lambda}_{\mu\nu}$ by the relation

$$R_{\mu\nu} = \frac{\partial \Gamma^{\sigma}_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial \Gamma^{\sigma}_{\mu\sigma}}{\partial x^{\nu}} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} \tag{6}$$

As in our previous works on general relativity, we assumed that the field equations of general relativity could also be used to describe spacetime structures of quantum particles. In particular, for the purpose of describing the spacetime dynamics of a single quantum particle, we considered a general relativistic spacetime model for a quantum particle based on the Robertson-Walker metric of the form [4]

$$ds^2 = c^2 dt^2 - S^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{7}$$

with $k = -1, 0, 1$. As shown in the appendix, these values of k represent the geometry of three-dimensional spaces of constant curvature in which the value of $k = 1$ corresponds to a three-dimensional space with positive curvature, such as a three-dimensional sphere in four-dimensional Euclidean space, the value of $k = 0$ corresponds to the three-dimensional Euclidean space and the value of $k = -1$ corresponds to a three-dimensional space with negative curvature. It is interesting to note that even though a three-dimensional space of a constant negative curvature cannot be embedded in a four-dimensional Euclidean space, it can be embedded in a flat Minkowski space with signature $+2$, such as a three-dimensional hyperboloid can be embedded in a four-dimensional Minkowski space [5]. Since the energy-momentum tensor has the unit of pressure, which may be viewed as an inverse square law, therefore it is reasonable to assume that the energy-momentum tensor takes the form

$$T_{\mu\nu} = \frac{A}{S^2(t)}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

where A is an undetermined constant and the time-dependent quantity $S(t)$ in the Robertson-Walker line element is considered to represent the radius of curvature of the spacetime manifold and will be determined by the field equations of general relativity. From the Robertson-Walker line element, the metric tensor $g_{\mu\nu}$ is written out as

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{S^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -S^2 r^2 & 0 \\ 0 & 0 & 0 & -S^2 r^2 \sin^2\theta \end{pmatrix} \tag{9}$$

If the affine connection $\Gamma^\lambda_{\mu\nu}$ is defined in terms of the metric tensor $g_{\mu\nu}$ as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right) \tag{10}$$

then the non-zero components of the affine connection $\Gamma^\lambda_{\mu\nu}$ are

$$\Gamma^0_{11} = \frac{S}{c(1-kr^2)}\frac{dS}{dt}, \qquad \Gamma^0_{22} = \frac{Sr^2}{c}\frac{dS}{dt}, \qquad \Gamma^0_{33} = \frac{Sr^2\sin^2\theta}{c}\frac{dS}{dt}$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{1}{cS}\frac{dS}{dt}, \quad \Gamma^1_{11} = \frac{kr}{1-kr^2}, \quad \Gamma^1_{22} = -r(1-kr^2), \quad \Gamma^1_{33} = -r\sin^2\theta(1-kr^2)$$

$$\Gamma^2_{02} = \Gamma^2_{20} = \frac{1}{cS}\frac{dS}{dt}, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{cS}\frac{dS}{dt}, \qquad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \tag{11}$$

Using the non-zero components of the affine connection $\Gamma^\lambda_{\mu\nu}$ given in Equation (11), the non-zero components of the Ricci tensor $R_{\mu\nu}$ given in Equation (6) are found

$$R_{00} = -\frac{3}{c^2 S}\frac{d^2S}{dt^2} \tag{12}$$

$$R_{11} = \frac{1}{(1-kr^2)}\left(\frac{S}{c^2}\frac{d^2S}{dt^2} + \frac{2}{c^2}\left(\frac{dS}{dt}\right)^2 + 2k\right) \tag{13}$$

$$R_{22} = \frac{r^2 S}{c^2}\frac{d^2S}{dt^2} + \frac{2r^2}{c^2}\left(\frac{dS}{dt}\right)^2 + 2kr^2 \tag{14}$$

$$R_{33} = \left(\frac{r^2 S}{c^2}\frac{d^2S}{dt^2} + \frac{2r^2}{c^2}\left(\frac{dS}{dt}\right)^2 + 2kr^2\right)\sin^2\theta \tag{15}$$

and the Ricci scalar curvature is calculated as

$$R = -\frac{6}{c^2 S}\frac{d^2S}{dt^2} - \frac{6}{c^2 S^2}\left(\frac{dS}{dt}\right)^2 - \frac{6k}{S^2} \tag{16}$$

From Einstein field equations given in Equation (5), we obtain the following system of equations for the quantity S

$$\frac{1}{S^2}\left(\frac{dS}{dt}\right)^2 + \frac{kc^2}{S^2} + \frac{\Lambda c^2}{3} = \frac{\kappa c^2}{3}\frac{A}{S^2} \tag{17}$$

$$\frac{2}{S}\frac{d^2S}{dt^2} + \frac{1}{S^2}\left(\frac{dS}{dt}\right)^2 + \frac{kc^2}{S^2} - \Lambda c^2 = 0 \tag{18}$$

This system of equations has a solution of the form

$$S = act \qquad \text{where} \qquad a = \sqrt{\frac{\kappa A}{4} - k} \tag{19}$$

In the following we will discuss three different cases for which $k = -1, 0$ and 1. Let us first consider the case when $k = 1$, which corresponds to spacetime structures of quantum particles with positive curvature. If we assume a reasonable value for the energy density so that $|\kappa A/4| \ll 1$, then we can quantise the quantum structure of the particle by letting $S = ict$. As shown below, the quantisation ansatz transforms the Robertson-Walker metric into a manifestly Minkowski metric. This kind of quantisation also turns the pseudo-Riemannian curved spacetime of a quantum particle into a Riemannian spacetime and the quantum particle is viewed to exist as a three-dimensional manifold embedded in the four-dimensional manifold. In order to transform the Robertson-Walker metric of the particle into a manifestly Minkowski metric, we apply the coordinate transformations [4]

$$iR = ctr, \qquad cT = ct\sqrt{1-r^2} \tag{20}$$

$$r = \left(1-\frac{c^2T^2}{R^2}\right)^{-1/2}, \qquad ct = iR\left(1-\frac{c^2T^2}{R^2}\right)^{1/2} \tag{21}$$

By using the transformation of the metric tensors

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta} \tag{22}$$

the coordinate transformations given in Equations (20) and (21) reduce the Robertson-Walker metric of the quantum particle to a manifestly Minkowski metric of the form

$$ds^2 = c^2dT^2 - dR^2 - R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{23}$$

We now show that the spacetime dynamics of the quantum particle can be formulated using an action integral. In terms of the coordinates (cT, R), the quantity $S = ict$ can be written as an action integral as follows

$$S = -i\sqrt{c^2T^2 - R^2} = -i\int ds = -ic\int\sqrt{1-\frac{v^2}{c^2}}dT \tag{24}$$

where ds is the manifestly Minkowski spacetime interval and $v = R/T$. It can be verified that the quantity S satisfies the following relation

$$\left(\frac{\partial S}{\partial R}\right)^2 - \frac{1}{c^2}\left(\frac{\partial S}{\partial T}\right)^2 - 1 = 0 \tag{25}$$

Following Schrödinger's method in wave mechanics [7], if we introduce a new quantity Ψ defined by the relation

$$S = K\ln\Psi \tag{26}$$

where K is a dimensional constant, then the quantity Ψ satisfies the equation

$$\left(\frac{\partial\Psi}{\partial R}\right)^2 - \frac{1}{c^2}\left(\frac{\partial\Psi}{\partial T}\right)^2 - \frac{1}{K^2}\Psi^2 = 0 \tag{27}$$

Applying the variational principle [8], Equation (27) can be reduced to a Klein-Gordon-like wave equation

$$\nabla^2\Psi - \frac{1}{\mathrm{c}_\alpha^2}\frac{\partial^2\Psi}{\partial \mathrm{T}^2} - \frac{1}{\mathrm{K}_\alpha^2}\Psi = 0 \tag{28}$$

where $c_\alpha = c/\sqrt{2}$ and $K_\alpha = K/\sqrt{2}$. Furthermore, a comparison between Equation (28) and the Klein-Gordon equation gives $K_\alpha = \hbar/mc_\alpha$. It is worth noting that even though the wavefunction Ψ obtained from the wave equation given in Equation (28) is expressed in terms of the coordinates (cT, R) in a manifestly Minkowski spacetime, by applying the coordinate transformations given in Equation (20) it can be transformed to a wavefunction in a curved spacetime with coordinates (ct, r). Furthermore, it is also observed that Equation (26) can be rewritten in the form

$$\Psi = e^{-\frac{i}{K}\int ds} \tag{29}$$

With this form we can recover standard quantum mechanics for quantum particles in the non-relativistic limit by applying the Feynman path integral method [6].

Next we consider the case when $k = -1$, which corresponds to spacetime structures of quantum particles with negative curvature. If we also assume $|\kappa A/4| \ll 1$ then we have $S = ct$. The coordinate transformations of the forms

$$R = ctr, \qquad cT = ct\sqrt{1+r^2} \tag{30}$$

$$r = \left(\frac{c^2T^2}{R^2} - 1\right)^{-1/2}, \qquad ct = R\left(\frac{c^2T^2}{R^2} - 1\right)^{1/2} \tag{31}$$

reduce the Robertson-Walker line element of a quantum particle to that of a manifestly Minkowski spacetime given in Equation (23). The action integral S in terms of the coordinates (cT, R) is given by

$$S = \sqrt{c^2T^2 - R^2} = \int ds = c\int\sqrt{1 - \frac{v^2}{c^2}}\,dT \tag{32}$$

where ds is the manifestly Minkowski spacetime interval and $v = R/T$. It can be verified that the quantity S satisfies the following relation

$$\left(\frac{\partial S}{\partial R}\right)^2 - \frac{1}{c^2}\left(\frac{\partial S}{\partial T}\right)^2 + 1 = 0 \tag{33}$$

If we now introduce a new quantity Ψ defined by the relation

$$S = iK\ln\Psi \tag{34}$$

then the quantity Ψ then satisfies the equation

$$\left(\frac{\partial \Psi}{\partial R}\right)^2 - \frac{1}{c^2}\left(\frac{\partial \Psi}{\partial T}\right)^2 - \frac{1}{K^2}\Psi^2 = 0 \tag{35}$$

The variational principle then reduces Equation (35) to a Klein-Gordon-like wave equation

$$\nabla^2\Psi - \frac{1}{c_\alpha^2}\frac{\partial^2\Psi}{\partial T^2} - \frac{1}{K_\alpha^2}\Psi = 0 \tag{36}$$

where $c_\alpha = c/\sqrt{2}$ and $K_\alpha = K/\sqrt{2}$.

Finally, let us first consider the case when $k = 0$, which corresponds to spacetime structures of quantum particles with zero curvature. A real solution is obtained by any positive energy density, $S = act$. The spatial part of the Robertson-Walker line element simply becomes the Euclidean metric scaled by the factor S. If we apply the coordinate transformations

$$R = actr, \qquad cT = ct \tag{37}$$

then $dR = actdr + acrdT$. It is seen that when the term $acrdT \ll 1$, the spacetime structure of a quantum particle can be reduced to that of the Minkowski spacetime. For a particle with a large energy density, $a \gg 1$, its curved spacetime metric can only be transformed to Minkowski metric for a short time $dT \ll 1$.

2. Euclidean General Relativistic Formulation of Quantum Mechanics

As shown in our previous works [9], Euclidean relativity can be formulated by considering the following modified Lorentz transformation

$$x' = \gamma_E(x - \beta ct) \tag{38}$$

$$y' = y \tag{39}$$

$$z' = z \tag{40}$$

$$ct' = \gamma_E(\beta x + ct) \tag{41}$$

where $\beta = v/c$ and γ_E will be determined from the principle of relativity and the postulate of a universal speed. Instead of assuming the invariance of the Minkowski spacetime interval, if we now assume the invariance of the Euclidean interval $c^2t^2 + x^2 + y^2 + z^2$ then from the modified Lorentz transformation given in Equations (38-41), we obtain the following expression for γ_E

$$\gamma_E = \frac{1}{\sqrt{1 + \beta^2}} \tag{42}$$

It is seen from the expression of γ_E given in Equation (42) that there is no upper limit in the relative speed v between inertial frames. The value of γ_E at the universal speed $v = c$ is $\gamma_E = 1/\sqrt{2}$. For the values of $v \ll c$, the modified Lorentz transformation given in Equations (38-41) also reduces to the Galilean transformation. However, it is interesting to observe that when $\beta \to \infty$ we have $\gamma_E \to 0$ and $\beta\gamma_E \to 1$, and in this case from Equations (38) and (41), we obtain $x' \rightarrow -ct$ and $ct' \rightarrow x$, respectively. This result shows that there is a conversion between space and time when $\beta \rightarrow \infty$, therefore in Euclidean special relativity, not only the concept of motion but the concepts of space and time themselves are also relative. It is also worth mentioning here that the Euclidean relativity of space and time also provides a profound foundation for the temporal dynamics that we have discussed in our other works [10]. In the present situation, if in the inertial frame S with spacetime coordinates (ct, x, y, z) the dynamics of a particle is described by Newton's second law $m\,d^2\mathbf{r}/dt^2 = \mathbf{F}$, then since $x' \rightarrow -ct$ and $ct' \rightarrow x$ it is seen that the spatial Newton's second law in the inertial frame S is converted to a temporal law of dynamics $D\,d^2\mathbf{t}/dr^2 = \mathbf{F}$ in the inertial frame S' with spacetime coordinates (ct', x', y', z'). As also discussed in our previous works on the field equations of general relativity [2], if we rewrite Einstein field equations in the following form

$$T_{\alpha\beta} = \frac{1}{\kappa}\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right) \tag{43}$$

then Einstein field equations can be interpreted as a definition of an energy-momentum tensor as that of Maxwell theory of the electromagnetic field. In this case, the basic equations of the gravitational field can be proposed using the contracted Bianchi identities

$$\nabla_\beta R^{\alpha\beta} = \frac{1}{2}g^{\alpha\beta}\nabla_\beta R \tag{44}$$

Even though Equation (44) is purely geometrical, it has a form of Maxwell field equations of the electromagnetic tensor, $\partial_\alpha F^{\alpha\beta} = \mu j^\beta$. If the quantity $\frac{1}{2}g^{\alpha\beta}\nabla_\beta R$ can be perceived as a physical entity, such as a four-current of gravitational matter, then Equation (44) has the status of a dynamical law of a physical theory. With the assumption that the quantity $\frac{1}{2}g^{\alpha\beta}\nabla_\beta R$ to be identified with a four-current of gravitational matter then a four-current $j^\alpha = (\rho, \mathbf{j}_i)$ can be defined purely geometrical as follows

$$j^\alpha = \frac{1}{2}g^{\alpha\beta}\nabla_\beta R \tag{45}$$

For a purely gravitational field, Equation (44) reduces to

$$\nabla_\beta R^{\alpha\beta} = 0 \tag{46}$$

Using the identity $\nabla_\mu g^{\alpha\beta} \equiv 0$, Equation (46) implies

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \tag{47}$$

where Λ is an undetermined constant. Using the identities $g_{\alpha\beta}g^{\alpha\beta} = 4$ and $g_{\alpha\beta}R^{\alpha\beta} = R$, we obtain $\Lambda = R/4$, and the energy-momentum tensor given in Equation (43) reduces to

$$T_{\alpha\beta} = -\frac{\Lambda}{\kappa}g_{\alpha\beta} \tag{48}$$

Now we consider how to formulate quantum mechanics from the Euclidean general relativity. As in the case of the pseudo-Euclidean general relativistic formulation of quantum mechanics, for the purpose of describing the spacetime dynamics of a single quantum particle, we also consider a general relativistic spacetime model for a quantum particle based on the Euclidean Robertson-Walker metric of the form

$$ds^2 = c^2dt^2 + S^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{49}$$

with $k = -1, 0, 1$. The quantity $S(t)$ is considered to represent the radius of curvature of the spacetime manifold and the metric tensor $g_{\mu\nu}$ now takes the form

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{S^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & S^2r^2 & 0 \\ 0 & 0 & 0 & S^2r^2\sin^2\theta \end{pmatrix} \tag{50}$$

The non-zero components of the affine connection $\Gamma^\lambda_{\mu\nu}$ are

$$\Gamma^0_{11} = -\frac{S}{c(1-kr^2)}\frac{dS}{dt}, \quad \Gamma^0_{22} = -\frac{Sr^2}{c}\frac{dS}{dt}, \quad \Gamma^0_{33} = -\frac{Sr^2\sin^2\theta}{c}\frac{dS}{dt}$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{1}{cS}\frac{dS}{dt}, \quad \Gamma^1_{11} = \frac{kr}{1-kr^2}, \quad \Gamma^1_{22} = -r(1-kr^2), \quad \Gamma^1_{33} = -r\sin^2\theta(1-kr^2)$$

$$\Gamma^2_{02} = \Gamma^2_{20} = \frac{1}{cS}\frac{dS}{dt}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{cS}\frac{dS}{dt}, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \tag{51}$$

The non-zero components of the Ricci tensor are

$$R_{00} = -\frac{3}{c^2S}\frac{d^2S}{dt^2} \tag{52}$$

$$R_{11} = \frac{1}{(1-kr^2)}\left(-\frac{S}{c^2}\frac{d^2S}{dt^2} - \frac{2}{c^2}\left(\frac{dS}{dt}\right)^2 + 2k\right) \tag{53}$$

$$R_{22} = -\frac{r^2S}{c^2}\frac{d^2S}{dt^2} - \frac{2r^2}{c^2}\left(\frac{dS}{dt}\right)^2 + 2kr^2 \tag{54}$$

$$R_{33} = \left(-\frac{r^2S}{c^2}\frac{d^2S}{dt^2} - \frac{2r^2}{c^2}\left(\frac{dS}{dt}\right)^2 + 2kr^2\right)\sin^2\theta \tag{55}$$

The Ricci scalar curvature is found as

$$R = -\frac{6}{c^2 S}\frac{d^2 S}{dt^2} - \frac{6}{c^2 S^2}\left(\frac{dS}{dt}\right)^2 + \frac{6k}{S^2} \tag{56}$$

From Einstein field equations given in Equation (5) and the energy-momentum tensor given in Equation (8), we have

$$\frac{1}{S^2}\left(\frac{dS}{dt}\right)^2 - \frac{kc^2}{S^2} + \frac{\Lambda c^2}{3} = \frac{\kappa c^2}{3}\frac{A}{S^2} \tag{57}$$

$$\frac{2}{S}\frac{d^2 S}{dt^2} + \frac{1}{S^2}\left(\frac{dS}{dt}\right)^2 - \frac{kc^2}{S^2} - \Lambda c^2 = 0 \tag{58}$$

This system of equations has a solution of the form

$$S = act, \qquad \text{where} \qquad a = \sqrt{\frac{\kappa A}{4} + k} \tag{59}$$

In the following we will also consider three different cases for $k = -1, 0$ and 1. Let us first consider the case when $k = 1$, which corresponds to spacetime structures of quantum particles of positive curvature. If we assume $|\kappa A/4| \ll 1$ then we have $S = ct$. The coordinate transformations of the forms

$$R = ctr, \qquad cT = ct\sqrt{1 - r^2} \tag{60}$$

$$r = \left(1 + \frac{c^2 T^2}{R^2}\right)^{-1/2}, \qquad ct = R\left(1 + \frac{c^2 T^2}{R^2}\right)^{1/2} \tag{61}$$

reduce the Euclidean Robertson-Walker line element of a quantum particle given in Equation (49) to that of a manifestly Euclidean spacetime

$$ds^2 = c^2 dT^2 + dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{62}$$

The action integral S in terms of the coordinates (cT, R) is

$$S = \sqrt{c^2 T^2 + R^2} = \int ds = c\int \sqrt{1 + \frac{v^2}{c^2}}\, dT \tag{63}$$

where ds is the manifestly Euclidean spacetime interval and $v = R/T$. It can be verified that the quantity S satisfies the following relation

$$\left(\frac{\partial S}{\partial R}\right)^2 + \frac{1}{c^2}\left(\frac{\partial S}{\partial T}\right)^2 - 1 = 0 \tag{64}$$

If we also introduce a new quantity Ψ defined by the relation $S = K\ln\Psi$, where K is a dimensional constant, then the quantity Ψ then satisfies the equation

$$\left(\frac{\partial\Psi}{\partial R}\right)^2 + \frac{1}{c^2}\left(\frac{\partial\Psi}{\partial T}\right)^2 - \frac{1}{K^2}\Psi^2 = 0 \tag{65}$$

Using the variational principle, Equation (65) can be reduced to the following equation

$$\nabla^2\Psi + \frac{1}{\mathrm{c}_\alpha^2}\frac{\partial^2\Psi}{\partial \mathrm{T}^2} - \frac{1}{\mathrm{K}_\alpha^2}\Psi = 0 \tag{66}$$

where $c_\alpha = c/\sqrt{2}$ and $K_\alpha = K/\sqrt{2}$. However, Equation (66) differs from the Klein-Gordon equation by the appearance of a plus sign in the temporal term. This results from the fact that quantum particles in this case are embedded in a Euclidean space.

Now consider the case $k = -1$, which corresponds to spacetime structures of quantum particles of negative curvature. If we also assume that $|\kappa A/4| \ll 1$, then we can quantise the quantum structure of the particle by letting $S = ict$. If we now apply the coordinate transformations

$$iR = ctr, \qquad cT = ct\sqrt{1+r^2} \tag{67}$$

$$r = i\left(1+\frac{c^2T^2}{R^2}\right)^{-1/2}, \qquad ct = R\left(1+\frac{c^2T^2}{R^2}\right)^{1/2} \tag{68}$$

then using the transformation of the metric tensor given in Equation (22) we can reduce the Euclidean Robertson-Walker metric of the quantum particle to a manifestly Euclidean metric of the form given in Equation (62). The spacetime dynamics of the quantum particle can be formulated using the action integral S in terms of the coordinates (cT, R) given by

$$S = i\sqrt{c^2T^2 + R^2} = i\int ds = ic\int\sqrt{1+\frac{v^2}{c^2}}\,dT \tag{69}$$

It can be verified that the quantity S satisfies the following relation

$$\left(\frac{\partial S}{\partial R}\right)^2 + \frac{1}{c^2}\left(\frac{\partial S}{\partial T}\right)^2 + 1 = 0 \tag{70}$$

If we also introduce a new quantity Ψ defined by the relation $S = iK\ln\Psi$, where K is a dimensional constant, then the quantity Ψ then satisfies the equation

$$\left(\frac{\partial\Psi}{\partial R}\right)^2 + \frac{1}{c^2}\left(\frac{\partial\Psi}{\partial T}\right)^2 - \frac{1}{K^2}\Psi^2 = 0 \tag{71}$$

Using the variational principle, Equation (71) is reduced to the following equation

$$\nabla^2\Psi + \frac{1}{\mathrm{c}_\alpha^2}\frac{\partial^2\Psi}{\partial \mathrm{T}^2} - \frac{1}{\mathrm{K}_\alpha^2}\Psi = 0 \tag{72}$$

where $c_\alpha = c/\sqrt{2}$ and $K_\alpha = K/\sqrt{2}$.

Finally, let us first consider the case when $k = 0$ which corresponds to particles with zero curvatures. A real solution is obtained as $S = act$. The spatial part of the Robertson-Walker line element simply becomes the Euclidean metric scaled by the factor S. If we apply the coordinate transformations

$$R = actr, \qquad cT = ct \tag{73}$$

then $dR = actdr + acrdT$. It is seen that when the term $acrdT \ll 1$ the spacetime structure of a quantum particle can be reduced to that of the Euclidean spacetime. For a particle with a large energy density, $a \gg 1$, its curved spacetime metric can only be transformed to Euclidean metric for a short time $dT \ll 1$.

Appendix

The three-dimensional surfaces of hyperspheres have constant positive curvature. A hypersphere of radius S can be embedded in a four-dimensional Euclidean space with the equation given in the Cartesian coordinates x_μ, $\mu = 1, 2, 3, 4$ as

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = S^2 \tag{1}$$

The spatial line element $d\sigma$ on the surface is

$$d\sigma^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \tag{2}$$

If we use the intrinsic coordinates defined by

$$x_1 = S\sin\chi\cos\theta \tag{3}$$

$$x_2 = S\sin\chi\sin\theta\cos\phi \tag{4}$$

$$x_3 = S\sin\chi\sin\theta\sin\phi \tag{5}$$

$$x_4 = S\cos\chi \tag{6}$$

then the line element is written as

$$d\sigma^2 = S^2\left(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{7}$$

If we define $r = \sin\chi$, then $dr = \cos\chi\, d\chi$ and $dr^2 = (1 - r^2)d\chi^2$, we finally obtain

$$d\sigma^2 = S^2\left(\frac{dr^2}{1 - r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{8}$$

The three-dimensional surfaces of constant negative curvature can be represented by the equation given in the Cartesian coordinates x_μ, $\mu = 1, 2, 3, 4$ as

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = -S^2 \tag{9}$$

The spatial line element $d\sigma$ on the surface is

$$d\sigma^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 \tag{10}$$

If we use the intrinsic coordinates defined by

$$x_1 = S\sinh\chi\cos\theta \tag{11}$$

$$x_2 = S\sinh\chi\sin\theta\cos\phi \tag{12}$$

$$x_3 = S\sinh\chi\sin\theta\sin\phi \tag{13}$$

$$x_4 = S\cosh\chi \tag{14}$$

then the line element is written as

$$d\sigma^2 = S^2\big(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\big) \tag{15}$$

If we define $r = \sinh\chi$, then $dr = \cosh\chi\, d\chi$ and $dr^2 = (1 + r^2)d\chi^2$, we finally obtain

$$d\sigma^2 = S^2\left(\frac{dr^2}{1 + r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{16}$$

It is noted that due to the negative sign in front of the dx_4^2 in Equation (10), the three-dimensional surfaces of constant negative curvature are not embedded in a four-dimensional Euclidean space but in a four-dimensional pseudo-Euclidean space.

The three-dimensional surfaces of zero curvature with the Euclidean line element scaled by a factor S can be written in the form

$$d\sigma^2 = S^2\big(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\big) \tag{17}$$

With the spatial line elements given in Equations (8), (16) and (17) and if the constant S depends on the cosmic time then the Robertson-Walker line element is obtained

$$ds^2 = c^2dt^2 - S^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{18}$$

References

[1] Vu B Ho, *On the motion of quantum particles and Euclidean relativity* (Preprint, ResearchGate, 2017), viXra 1710.0253v1.

[2] Vu B Ho, *Spacetime Structures of Quantum Particles* (Preprint, ResearchGate, 2017), viXra 1708.0192v1.

[3] Einstein, A. *et al*, *The Principle of Relativity*, Dover, New York, 1952.

[4] J. V. Narlikar, *Introduction to Cosmology*, Cambridge University Press, Cambridge, 1993.

[5] Ray D'Inverno, *Introducing Einstein' Relativity*, Clarendon Press, Oxford, 1992.

[6] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965.

[7] Erwin Schrödinger, *Collected Papers on Wave Mechanics* (AMS Chelsea Publishing, New York, 1982).

[8] W. Yourgrau and S. Mandelstam, *Variational Principles in Dynamics and Quantum Theory*, Dover Publications, New York, 1979.

[9] Vu B Ho, *Euclidean Relativity* (Preprint, ResearchGate, 2017), viXra 1710.0302v1.

[10] Vu B Ho, *A Temporal Dynamics: A Generalised Newtonian and Wave Mechanics* (Preprint, ResearchGate, 2016), viXra 1708.0198v1.