A Simple Proof that $\zeta(n \geq 2)$ is Irrational

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Abstract

We prove that partial sums of $\zeta(n) - 1 = z_n$ are not given by any single decimal in a number base given by a denominator of their terms. This result, applied to all partials, shows that partials are excluded from an ever greater number of rational, possible convergence points. The limit of the partials is z_n and the limit of the exclusions leaves only irrational numbers. Thus z_n is proven to be irrational.

1 Introduction

Apery's $\zeta(3)$ proof is the only proof that a specific odd argument for $\zeta(n)$ is irrational. Even arguments are a natural consequence of Bernoulli formula [2] for $\zeta(2n)$.

Beuker, based on the work of Apery, gives a proof that $\zeta(2)$ is irrational [3]. These proofs for $\zeta(2)$ and $\zeta(3)$ require the prime number theorem, as well as subtle $\epsilon - \delta$ reasoning. The puzzle is, then, if you can use Apery's idea for an easier, as it turns out case, that of $\zeta(2)$, why can't you generalize Apery's idea to the general $\zeta(n)$ cases? Both the evens and odds?

Proving the general case using Apery's [1] central idea seems hopelessly elusive. It is not for a lack of trying. Apery's and other ideas can be seen in the very difficult results of Rivoal and Zudilin [7, 10]. Their results, that there are an infinite number of odd n such that $\zeta(n)$ is irrational and at least one of the cases 5,7,9, 11 likewise irrational, seem less than encouraging.

Another idea comes from Sondow's very easy *geometric* proof of the irrationality of e [9]. This proof uses what could be called an *eliminate as you define* idea. You build a number by a geometric process that eliminates other numbers (other rational numbers) from being possible convergence points. In the case of e there is a clear and easy connection between terms. Each term is a proportion of the previous and moves and squeezes partials from the left and right in a neat, orderly fashion. Trying the same trick with $\zeta(2)$, the right boundary doesn't necessarily contract in from a single boundary. Here's the key: it doesn't contract from a *single* boundary, but it does from a set of boundaries. For $\zeta(2)$, for example, $1/4 + 1/9$ is neither at $.x$ base 4 or $.y$ base 9. Continuing $1/4+1/9+1/16+1/5^2+1/6^2+1/7^2 = 282/551$ which is between 1/4s and 1/9s, but not between, any more, 1/4 and 1/2 and 3/9 and 4/9; it has blown passed these multiples of 1/4 and 1/9, but is still between some such multiples – implying not equal to any such multiples! In fact: in any number base, as the base is used to express a series, eventually decimal digits become fixed. For $\zeta(2) - 1$ in base 10, we can say $.6 < \zeta(2) - 1 < .7$. In another base, larger than 10, similar boundaries, fixed, will have to exist for sufficiently large upper limits of partials and will have to move in from .6 and .7, base 10. This is the idea we pursue in this paper.

This is an open number theory problem, so, for those that like challenges, I'll give here a sequence of problems to solve. I.e. see if you can do it before you read about how it was done. We will need two definitions:

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

Show that every rational number in $(0, 1)$ can be written as a single decimal using the denominators of the terms of any z_n. Next show the partial sums, s_k^n, can't be expressed as a single decimal in any of the terms of s_k^n. This implies that the precision of s_k^n increases. This is unlike something simple like $1/4+1/4 = 1/2$ – the precision or fineness of terms is 1/4 and that of the sum is 1/2, less precision – wider decimal intervals: base 2 versus base 4. Note that if a series converges to a rational number, its partials will get close to a number of less precision, in this sense. For example, $.\overline{1}$ base 4 converges to .1 base 3. So, having shown the denominators of the terms cover all rational possible convergence points and that the partials escape their terms, show that the partials can't converge to a number with finite precision and hence must converge to an irrational number.

2 Terms cover rationals

We start with something relatively easy.

Definition 1. *A decimal set, base j^n, is defined by*

$$D_{j^n} = \{1/j^n, \ldots, (j^n - 1)/j^n\} = \{.1, \ldots, .(j^n - 1)\} \text{ base } j^n.$$

That is D_{j^n} consists of all single decimals greater than 0 and less than 1 in base j^n.

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

Lemma 1.

$$\lim_{k\to\infty} \Xi_k^n = \bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0,1)$$

Proof. Every rational $a/b \in (0,1)$ is included in at least one D_{j^n}. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in D_{b^n}$. □

Note: Sondow's e is irrational proof gives this same idea. To wit, given a rational $0 < p/q < 1$

$$\frac{p}{q} = \frac{p(q-1)!}{q!}.$$

That is the denominators of the terms of e taken as number bases express all rational numbers in $(0,1)$.

3 Partials escape terms

Our aim in this section is to show that the reduced fractions that give the partial sums of z_n require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums of z_n can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms.

We will use z_2 to motivate the development. The partials of z_2, as they include all even k^2 in their denominators, will have a reduced form that has a greater power of two in the partial's denominator. This result is given in Lemma 2; it is similar to Apostol's chapter 1, problem 30. See [5] for a solution to this problem. Next, if we can show that there is at least one prime that does not recur in the k^2 denominators, then that prime will occur in the partial sum's reduced fraction. This result is given in Lemma 3. Such a prime does exist: Lemma 4, Bertrand's postulate.

The idea is simple. Consider $1/4 + 1/9 + 1/16 + 1/25$. There will be a power of 2 and of a relatively large prime in the denominator of the reduced sum. Indeed, the sum is 1669/3600 and the denominator of this reduced form has the prime factorization of $2^4 3^2 5^2$; it has relatively large power of 2 and prime 5. The prime is between 3 and 6 as Bertrand's postulate stipulates. As twice this prime exceeds the largest denominator in this partial sum, the partial sum can't be expressed as a single decimal in any of the denominators of the terms of the partial.

Lemma 2. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{1}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (1) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. $\square$

Lemma 3. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k,p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^n}{(k!)^n}\sum_{j=2}^{k}\frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{2}$$

As $(k,p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. $\square$

Lemma 4. *For any $k \geq 2$, there exists a prime p such that $k < p < 2k$.*

Proof. This is Bertrand's postulate [4]. $\square$

Theorem 1. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$.*

Proof. Using Lemma 4, for even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Lemma 4, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. $\square$

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

Proof. This is a restatement of Theorem 1. $\square$

One can get a geometric like idea similar to Sondow's. The partial s_k^n resides between decimal points in all the decimal sets in Ξ_k^n. Unlike the case of e, the intervals don't nest neatly. In fact, they migrate and overlap. Consider that z_2 has partials in the interval $[1/4, 2/4]$, but as $z_2 = .6\cdots > 2/4$, partials don't stay in this interval. But they do stay in some interval of the form

$[.(x-1), .x]$ of D_4. Although D_4 and D_{16} overlap, in this sense, s_k^2 will not be at any endpoint of D_{16}.

What happens when decimals become fixed? In every base they will become fixed. Eventually they all nest and like Sondow's e proof get trapped between all possible rational convergence points.

4 A Suggestive Table

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\dots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\dots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 1: A list of all rational numbers between 0 and 1 is given by the number sets along the diagonal. Partials of z_2 are excluded from sets below and to the upper left of the partial.

The result of applying Corollary 1 to all partial sums of z_2 is given in Table 1.[1] The table shows that adding the numbers above each D_{k^2}, for all $k \geq 2$ gives results not in D_{k^2} or any previous rows' such sets. So, for example, $1/4+1/9$ is not in D_4, $1/4+1/9$ is not in D_4 or D_9, $1/4+1/9+1/16$ is not in D_4, D_9, or D_{16}, etc.. That's what Corollary 1 says.

Lemma 1 says that for all the series z_n the denominators of their terms *cover* the possible rational convergence points and Corollary 1 says the partial sums of z_n *escape* their terms. As all rational numbers between 0 and 1 are

[1] Table 1 might remind readers of Cantor's diagonal method. We don't pursue this idea in this article. See [6].

in Ξ_k^n for some k sufficiently large this says partials are being, so to speak, chased out of the Ξ_k^n park – possible rational convergence points. Where could they go but to the irrational zoo, sorry!

5 A Simple Proof

We will designate the set of rational numbers in $(0,1)$ with $\mathbb{Q}(0,1)$, the set of irrationals in $(0,1)$ with $\mathbb{H}(0,1)$, and the set of real numbers in $(0,1)$ with $\mathbb{R}(0,1)$. We use $\mathbb{R}(0,1) = \mathbb{Q}(0,1) \cup \mathbb{H}(0,1)$ and $\mathbb{Q}(0,1) \cap \mathbb{H}(0,1) = \emptyset$ in the following.

Theorem 2. *z_n is irrational.*

Proof. Corollary 1 implies $s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n$. As $\lim_{k\to\infty} s_k^n = z_n$, using Lemma 3, we have

$$z_n \in \mathbb{R}(0,1) \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1). \tag{3}$$

That is z_n is irrational. □

Some mathematicians, including the author, find this proof unsatisfactory, but it is simple. I suggest that mathematicians have been trained to be unduly suspicious of proofs like the above. Consider that the above is just the same as

$$\lim_{n\to\infty}(-1/n, 1/n) = \emptyset,$$

where $(-1/n, 1/n)$ designates an open interval. I'm with you in wanting to viscerally feel a squeeze at work, so here's another proof.

6 Towards greater precision

We drop the n subscript used previously with z_n and slightly modify such use of subscripts and superscripts in this section. The context should make meanings clear. In that regard, we use the bases k as a fill in for base k^n, to further simplify notation.

Definition 3. *Let $D_k^{\epsilon_k}$ be the set of all D_k decimal sets having an element within ϵ_k of s_k.*

Lemma 5. *Let z be the convergence point of the series with partials s_k. Then z is irrational if there exists a monotonically decreasing sequence ϵ_k such that*

$$\lim_{k\to\infty} \epsilon_k = 0,$$

and

$$\bigcap_{k=2}^{\infty} D_k^{\epsilon_k} = \emptyset. \tag{4}$$

Proof. We use proof by contraposition: $p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$. Suppose z is rational then $z \in D_k^*$, a specific decimal set. Define

$$\epsilon_m = z - s_m$$

and set

$$\epsilon_k = 2\epsilon_m.$$

Then

$$D_k^* \subset \bigcap_{k=2}^{\infty} D_k^{\epsilon_k},$$

so the intersection is not empty. □

Definition 4. *The precision of a decimal base, b, is $1/b$.*

Lemma 6. *Given any ϵ there exists a decimal base b of greater precision than ϵ; that is*

$$\frac{1}{b} < \epsilon.$$

Proof. This is the Archimedean property of the reals [8]. □

Theorem 3. *z_n is irrational.*

Proof. We need to define a sequence ϵ_k. Let

$$\epsilon_k^* = \min\{|x - s_k| : x \in \Xi_2^k\}.$$

We know by Corollary 1 that $\epsilon_k^* > 0$. We proceed inductively. For the first iteration, let ϵ_3 be a number such that $\epsilon_3 < \epsilon_3^*$. This excludes the decimal sets of Ξ_2^3 at this our first iteration. Assume we can generally do this for the kth iteration. For the $k+1$st iteration, using Lemma 6, there exists a base in Ξ_2^{k+r}, for some r such that $\epsilon^*_{k+r} < \epsilon_k/2$. Set $\epsilon_{k+1} = \epsilon^*_{k+r}$. The procedure gives ϵ values that exclude ever more decimal sets from $D_k^{\epsilon_k}$. Regroup the series. By Lemma 1, the exclusions are exhaustive, so

$$\bigcap_{k=2}^{\infty} D_k^{\epsilon_k} = \emptyset,$$

as needed. □

Conclusion

The alternate proof definitely has a squeeze action to it and is more satisfying. It seems like Sondow's e proof with epsilon reasoning added.

Now that you are loaded up with all these decimal sets and the like, you should be able to comprehend a yet shorter and more Sondow like (as in geometric) proof. For any interval $[.(y-1), .y]$ containing z_2 in some base b, there is an interval of the form $[.(x-1), .x]$ of D_r where r is the denominator of some distant (r much greater than b) s_k^2. Now suppose this $.(x-1)$ decimal is fixed in the approximation of z_2 in base r. This means $z \in [.(x-1), .x]$ but $.(y-1)$ and $.y$ are not equal to either $.(x-1)$ or $.x$ because they are in Ξ_k^2 and, per Corollary 1, $s_k^2 \notin \Xi_k^2$. So $.y$ can't be z_2. But this excludes all candidate rational numbers. This is a translation into math of the idea that an irrational number goes towards rational numbers of ever greater precision. Once you have a fixed decimal in a base there is an interval containing z, but there is a greater base of arbitrary precision yielding a similar interval nested inside. This is like the picture of Sondow's e proof. Greater precision is needed for better approximations.

One more angle of interest. Consider $.\overline{1}$ base 4. The partials all have finite decimals when bases are powers of 4. That is there exists a sequence of bases that give exact values of partials – no intervals. The equivalents for z are partials with bases the denominators of s_k. But the prime factors of these bases vary – new primes are introduced via Bertrand's postulate of Theorem 1. Now assume z is rational, say p/q, after a large upper limit of the partial the decimal approximation in base q should be of the form $.x\overline{q-1}$. But this approximates only one number. It is not possible for this to approximate the infinite number of different numbers given by s_k values. The $\overline{q-1}$ must stop and different fixed decimals must occur. Otherwise the uniqueness of decimal representations is violated. We have then a contradiction. Your finite alphabet won't spell an infinite number of words – given you are just repeating one letter!

By contradiction. Say $z = p/q$, then in base q the decimal representation of z must be $.(p-1)\overline{q-1}$. Now represent s_k values using base q. These have primes not shared by q and there are an infinite number of such partial sums. The representation of z in base q is impossible.

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