Simple Proofs that $\zeta(n \geq 2)$ is Irrational

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Abstract

We prove that partial sums of $\zeta(n) - 1 = z_n$ are not given by any single decimal in a number base given by a denominator of their terms. This result, applied to all partials, shows that partials are excluded from an ever greater number of rational, possible convergence points. The limit of the partials is z_n and the limit of the exclusions leaves only irrational numbers. Thus z_n is proven to be irrational. An alternative proof using Niven's *Irrational Numbers* is also given.

1 Introduction

Beuker gives a proof that $\zeta(2)$ is irrational [3]. It is calculus based, but requires the prime number theorem, as well as subtle $\epsilon - \delta$ reasoning. It generalizes only to the $\zeta(3)$ case. Here we give a simpler proof that uses just basic number theory (the easier chapters of Apostol and Hardy, [2, 4]) and treats all cases at once.

We use the following notation: for integers n, $n > 1$,

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

2 Decimals using denominators

Our aim in this section is to show that the reduced fractions that give the partial sums of z_n require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums of z_n can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms.

We will use z_2 to motivate the development. The partials of z_2, as they include all even k^2 in their denominators, will have a reduced form that has a greater power of two in the partial's denominator. This is result is given in Lemma 1; it is similar to Apostol's chapter 1, problem 30. See [5] for a solution to this problem. Next, if we can show that there is at least one prime that does not recur in the k^2 denominators, then that prime will occur in the partial sum's reduced fraction. This result is given in Lemma 2. We use Bertrand's postulate to show such a prime does exist in Lemma 3.

The idea is simple. Consider $1/4 + 1/9 + 1/16 + 1/25$. There will be a power of 2 and of a relatively large prime in the denominator of the reduced sum. Indeed, the sum is 1669/3600 and the denominator of this reduced form has the prime factorization of $2^4 3^2 5^2$; it has relatively large power of 2 and prime 5. The prime is between 3 and 6 as Bertrand's postulate stipulates. As twice this prime exceeds the largest denominator in this partial sum, the partial sum can't be expressed as a single decimal in any of the denominators of the terms of the partial.

Lemma 1. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{1}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (1) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. $\square$

Lemma 2. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k,p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{2}$$

As $(k,p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. $\square$

Theorem 1. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$.*

Proof. Bertrand's postulate states that for any $k \geq 2$, there exists a prime p such that $k < p < 2k$ [4]. For even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Bertrand's postulate, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. $\square$

In light of this result we give the following definitions and corollary.

Definition 1.

$$D_{j^n} = \{0, 1/j^n, \ldots, (j^n-1)/j^n\} = \{0, .1, \ldots, .(j^n-1)\} \textit{ base } j^n$$

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

Proof. Reduced fractions are unique. Suppose, to obtain a contradiction, that there exists $a/b \in \Xi_k^n$ such that $a/b = r/s$ then $b < s$ by Theorem 1. If a/b is not reduced, reduce it: $a/b = a_1/b_1$. A reduced fraction must have a smaller denominator than the unreduced form so $b_1 \leq b < s$ and this contradicts the uniqueness of the denominator of a reduced fraction. $\square$

3 A Suggestive Table

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\ldots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\ldots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 1: A list of all rational numbers between 0 and 1 is given by the number sets along the diagonal. Partials of z_2 are excluded from sets below and to the upper left of the partial.

The result of applying Corollary 1 to all partial sums of z_2 is given in Table 1.[1] The table shows that adding the numbers above each D_{k^2}, for all $k \geq 2$ gives results not in D_{k^2} or any previous rows' such sets. So, for example, $1/4+1/9$ is not in D_4, $1/4+1/9$ is not in D_4 or D_9, $1/4+1/9+1/16$ is not in D_4, D_9, or D_{16}, etc.. That's what Corollary 1 says.

[1] Table 1 might remind readers of Cantor's diagonal method. We don't pursue this idea in this article. See [7].

Lemma 3.

$$\lim_{k\to\infty} \Xi_k^n = \bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0,1)$$

Proof. Every rational $a/b \in (0,1)$ is included in at least one D_{j^n}. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in D_{b^n}$. □

Loosely speaking, Lemma 3 says that for all the series z_n the denominators of their terms *cover* the possible rational convergence points and Corollary 1 says the partial sums of z_n *escape* their terms.

4 Proof

We will designate the set of rational numbers in $(0,1)$ with $\mathbb{Q}(0,1)$, the set of irrationals in $(0,1)$ with $\mathbb{H}(0,1)$, and the set of real numbers in $(0,1)$ with $\mathbb{R}(0,1)$. We use $\mathbb{R}(0,1) = \mathbb{Q}(0,1) \cup \mathbb{H}(0,1)$ and $\mathbb{Q}(0,1) \cap \mathbb{H}(0,1) = \emptyset$ in the following.

Theorem 2. *z_n is irrational.*

Proof. Idea: Corollary 1 implies $s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n$. As $\lim_{k\to\infty} s_k^n = z_n$, using Lemma 3, we have

$$z_n \in \mathbb{R}(0,1) \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1). \tag{3}$$

That is z_n is irrational. □

Some mathematicians reject this proof, so here is another proof that follows easily as another consequence of Corollary 1.

Theorem 3. *z_n is irrational.*

Proof. As $\lim_{k\to\infty} s_k^n = z_n$, given any ϵ we can find an N_ϵ^n such that if $j > N_\epsilon^n$ then

$$z_n - s_j^n = \frac{x_j^n}{d_j^n}, \tag{4}$$

where d_j^n is the denominator of s_j^n and $0 < x_j^n < 1/d_j^n$. We know by Corollary 1 that (4) is never 0; (4) is based on decimal approximations of z_n in base d_j^n.

By convergence, x_j^n goes to 0. This gives infinitely many integers of the form

$$d_j^n z_n - d_j^n s_j^n = x_j^n. \tag{5}$$

That is there exists infinitely many integers d_j^n and $d_j^n s_j^n$ such that (5) holds. By Niven's Theorem 4.3 [8], this shows z_n is irrational. □

Conclusion

It is worth remarking that the proof given here seems the only one that works. Other types of proofs seem to get bogged down. One such is a *squeeze* action proof by Sondow for e's irrationality; see [6]. We note that $\mathbb{R}(0,1)\backslash\Xi_k^n$ consists of a union of open intervals with rational endpoints given by elements of Ξ_k^n and this is similar to the situation of e as developed in Sondow's paper. The catch is the intervals are much more complex than those for e; with these migrating and overlapping intervals distances to endpoints of partial sums can suddenly get very close to an endpoint and throw a wrench in works. Yet another strategy is that of Cantor's diagonal method (CDM) [7]. Like the proof given here, the central theme of applying CDM is to *define as you eliminate simultaneously.* The proof here combines Sondow with some epsilon-delta idea to get the job done. Dedekind cuts are a third idea. We note that Sondow's proof could be given via referencing Dedekind cuts. But alas without the combination of eliminate as you define with an epsilon delta idea, the approach gets stymied fatally.

Speaking of epsilon-delta proofs, proving the general case using Apery's [1] central idea (Apery showed $\zeta(3)$ is irrational) seems hopelessly elusive [9, 12]. Perhaps this is so because the combinatorial possibilities skyrocket with increasing n in $\zeta(n)$; and the strategy of epsilon-delta proofs needs the *eliminate as you define* key mentioned. Studying Apery's proof and Beukers simplifications of it the techniques for $\zeta(2)$ are mimicked for $\zeta(3)$ but are more complicated. One suspects, with increasing n, modifications of these two cases will be necessary and that the mechanics will grow too cumbersome. It is not for a lack of trying. One sees reminders of Apery's idea in the very difficult results of Rivoal and Zudilin [9, 12]; their results, that there are an infinite number of n such that $\zeta(n)$ is irrational and at least one of the cases 5,7,9, 11 are irrational, are less than encouraging.

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