A Simple Proof that $\zeta(n \geq 2)$ is Irrational

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Abstract

We prove that partial sums of $\zeta(n) - 1 = z_n$ are not given by any single decimal in a number base given by a denominator of their terms. This result, applied to all partials, shows that partials are excluded from an ever greater number of rational, possible convergence points. The limit of the partials is z_n and the limit of the exclusions leaves only irrational numbers. Thus z_n is proven to be irrational.

1 Introduction

Beuker gives a proof that $\zeta(2)$ is irrational [3]. It is calculus based, but requires the prime number theorem, as well as subtle $\epsilon - \delta$ reasoning. It generalizes only to the $\zeta(3)$ case. Here we give a simpler proof that uses just basic number theory (the easier chapters of Apostol and Hardy, [2, 4]) and treats all cases at once.

We use the following notation: for integers n, $n > 1$,

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

2 Decimals using denominators

Our aim in this section is to show that the reduced fractions that give the partial sums of z_n require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums of z_n can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms. Lemma 1 is similar to Apostol's chapter 1, problem 30. See [5] for a solution to this problem.

Lemma 1. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{1}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (1) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. □

Lemma 2. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{2}$$

As $(k, p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. □

Theorem 1. *If* $s_k^n = \frac{r}{s}$*, with* r/s *reduced, then* $s > k^n$*.*

Proof. Bertrand's postulate states that for any $k \geq 2$, there exists a prime p such that $k < p < 2k$ [4]. For even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Bertrand's postulate, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. □

In light of this result we give the following definitions and corollary.

Definition 1.

$$D_{j^n} = \{0, 1/j^n, \ldots, (j^n-1)/j^n\} = \{0, .1, \ldots, .(j^n-1)\} \text{ } \textit{base } j^n$$

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

Proof. Reduced fractions are unique. Suppose, to obtain a contradiction, that there exists $a/b \in \Xi_k^n$ such that $a/b = r/s$ then $b < s$ by Theorem 1. If a/b is not reduced, reduce it: $a/b = a_1/b_1$. A reduced fraction must have a smaller denominator than the unreduced form so $b_1 \leq b < s$ and this contradicts the uniqueness of the denominator of a reduced fraction. □

3 A Suggestive Table

The result of applying Corollary 1 to all partial sums of z_2 is given in Table 1.[1] The table shows that adding the numbers above each D_{k^2}, for all $k \geq 2$ gives results not in D_{k^2} or any previous rows' such sets. So, for example, $1/4 + 1/9$ is not in D_4, $1/4 + 1/9$ is not in D_4 or D_9, $1/4 + 1/9 + 1/16$ is not in D_4, D_9, or D_{16}, etc.. That's what Corollary 1 says.

[1] Table 1 might remind readers of Cantor's diagonal method. We don't pursue this idea in this article. See [7].

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\dots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\dots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 1: A list of all rational numbers between 0 and 1 is given by the number sets along the diagonal. Partials of z_2 are excluded from sets below and to the upper left of the partial.

Lemma 3.

$$\lim_{k\to\infty} \Xi_k^n = \bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0,1)$$

Proof. Every rational $a/b \in (0,1)$ is included in at least one D_{j^n}. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in D_{b^n}$. $\square$

Loosely speaking, Lemma 3 says that for all the series z_n the denominators of their terms *cover* the possible rational convergence points and Corollary 1 says the partial sums of z_n *escape* their terms.

4 Proof

We will designate the set of rational numbers in $(0,1)$ with $\mathbb{Q}(0,1)$, the set of irrationals in $(0,1)$ with $\mathbb{H}(0,1)$, and the set of real numbers in $(0,1)$ with $\mathbb{R}(0,1)$. We use $\mathbb{R}(0,1) = \mathbb{Q}(0,1) \cup \mathbb{H}(0,1)$ and $\mathbb{Q}(0,1) \cap \mathbb{H}(0,1) = \emptyset$ in the following.

Lemma 4. $\mathbb{R}(0,1) \setminus \Xi_k^n$ *consists of a union of open intervals with rational endpoints given by elements of* Ξ_k^n.

Proof. All cases will be the same. We will use z_2. Let

$$I_1 = \mathbb{R}(0,1) \setminus D_4. \tag{3}$$

This gives $I_1 = (0, 1/4) \cup (1/4, 2/4) \cup (2/4, 3/4) \cup (3/4, 1)$. This is $(0,1)$ with rational points of the form $x/4$ with $x = 1$, 2, and 3 removed. Now let

$$I_2 = \mathbb{R}(0,1) \setminus D_4 \cup D_9.$$

When the fractions are sorted in ascending order they are

$$\frac{1}{9}, \frac{2}{9}, \frac{1}{4}, \frac{3}{9}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{6}{9}, \frac{3}{4}, \frac{7}{9}, \frac{8}{9},$$

so

$$I_2 = (0, 1/9) \cup (1/9, 2/9) \cup (2/9, 1/4) \cup (1/4, 3/9) \cup (3/9, 4/9) \text{ and so on.}$$

□

Theorem 2. z_n *is irrational.*

Proof. Idea: Corollary 1 implies $s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n$. As $\lim_{k\to\infty} s_k^n = z_n$, using Lemma 3, we have

$$z_n \in \mathbb{R}(0,1) \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1). \tag{4}$$

That is z_n is irrational.

Details: Given any denominator of the form q, we can observe that there is a numerator x such that for all $r > R$

$$\frac{x-1}{q} < s_r^n < \frac{x}{q}. \tag{5}$$

This follows from $\mathbb{R}(0,1) \setminus \Xi_k^n$ consists of a union of open intervals with rational endpoints given by elements of Ξ_k^n, Lemma 4. Endpoints of the form $\frac{x}{q}$ will occur for some k in Ξ_k^n by Lemma 3. Corollary 1 implies that s_r^n in not being such an endpoint, s_r^n must be in an interval with such endpoints.

We claim z_n defines a Dedekind cut [9] for an irrational number. To show this we must show z_n is defined by two sets A and B; such that $A \cup B =$

$\mathbb{Q}(0,1)$; $A \cap B = \emptyset$; and every element of A is less than every element of B. This is best visualized as a cut in the $(0,1)$ segment of the real line with all rationals on one side of the cut or the other.

Given $\frac{p}{q} \in \mathbb{Q}(0,1)$, using (5), it must be that

$$s_k^n > \frac{p}{q} \text{ or } s_k^n < \frac{p}{q}, \tag{6}$$

for all k greater than some $K_{\frac{p}{q}}$. Therefore, for any $\frac{p}{q} \in \mathbb{Q}(0,1)$, eventually (6) holds. Thus z_n, as the limiting case, defines a Dedekind cut for an irrational number and must be irrational.

□

Conclusion

There is a *squeeze* action in the proof given for the irrationality of z_n. The proof adds a twist to the *squeeze* proof given by Sondow for e's irrationality. Whereas in Sondow's proof single intervals, like $[0, \frac{1}{2}]$ and $[\frac{3}{6}, \frac{4}{6}]$ (see [6]) are eliminated from locales of the convergence point of the series for e, now generic intervals of the form $(\frac{x-1}{q}, \frac{x}{q})$ are eliminated. The trick is to notice that if a partial occurs in one such interval, the partial is not equal to any endpoint with a denominator of q. With e and Sondow's proof for its irrationality, intervals containing partials are static, with the proof for z_n given here the intervals migrate, overlap and are more complicated.

Sondow's proof is an application of point-set topology [9, problem 21, page 82]. It could be slightly simplified (arguably) using, as we do here, Dedekind cuts. In both cases, then, we show a number is irrational by squeezing it between all plausible rational convergence points.

Could the proof given here be modified to use Sondow's original point-set topology proof? This gets to the nut of why this is (was, if I'm right) an unsolved number theory problem. Can we take contracting not always nesting intervals of the form $[\frac{x-1}{p}, \frac{x}{p}]$ with varying x? and apply Rudin's problem 21? No, not easily for sure, I suspect – at least not yet.

A theory might evolve that considers fractional moduli, things like

$$.5238 \equiv .0238 \bmod \frac{1}{4}$$

and

$$.5 \equiv 0 \bmod \frac{1}{4}.$$

The question becomes whether or not a sequence of shrinking intervals, not necessarily nested, but with endpoints that are not equivalent to 0 modulo $\frac{1}{n^2}$, have an intersection consisting of one irrational number? Using Dedekind cuts, as was done in this paper, the answer seems to be *yes*.

The proof given here seems simple, and, frankly, not particularly elegant. But proving the general case using Apery's [1] central idea (Apery showed $\zeta(3)$ is irrational) seems elusive [8, 11]. Perhaps this is so because the combinatorial possibilities skyrocket with increasing n in $\zeta(n)$; and the strategy of epsilon-delta proofs needs some relatively large gap to emerge to possible rational convergence points. Studying Apery's proof and Beukers presentation of the technique applied also to $\zeta(2)$ shows the mechanics for $\zeta(2)$ and $\zeta(3)$ just are impossible for general n. One suspects, with increasing n, these gaps grow too small, the mechanics too cumbersome and the techniques really are futile. A careful study of the very difficult results of Rivoal and Zudilin one sees reminders of Apery's idea [8, 11]; their results, that there are an infinite number of n such that $\zeta(n)$ is irrational and at least one of the cases 5,7,9, 11 are irrational are less than encouraging. What one really would like is a proof like Sondow's for e using point-set topology; nice and neat and obvious.

Yet another strategy is that of Cantor's diagonal method (CDM) [7]. Like the proof given here, the central theme of applying CDM is *define as you eliminate simultaneously*. This gets around, in my opinion, the problem of epsilon-delta proofs mentioned. With these migrating and overlapping intervals distances to endpoints of partial sums can suddenly get very close to an endpoint and throw a wrench in the epsilon-delta world. Dedekind cuts as applied in this article and CDM give a process of eliminating all possible rationals. They use progressively finer sieves rather than a single yard stick to find the gold. For very fine results for such numbers as $\zeta(n)$, you need sieves.

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