# A Simple Proof that $\zeta(n \geq 2)$ is Irrational

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**Abstract**

We prove that partial sums of $\zeta(n) - 1 = z_n$ are not given by any single decimal in a number base given by a denominator of their terms. This result, applied to all partials, shows that partials are excluded from an ever greater number of rational, possible convergence points. The limit of the partials is $z_n$ and the limit of the exclusions leaves only irrational numbers. Thus $z_n$ is proven to be irrational.

## 1 Introduction

Beuker gives a proof that $\zeta(2)$ is irrational [2]. It is calculus based, but requires the prime number theorem, as well as subtle $\epsilon - \delta$ reasoning. It generalizes only to the $\zeta(3)$ case. Here we give a simpler proof that uses just basic number theory (early chapters of Apostol and Hardy, [1, 3]) and treats all cases at once.

We use the following notation: for positive integers $n > 1$,

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

## 2 Decimals using denominators

Our aim in this section is to show that the reduced fractions that give the partial sums of $z_n$ require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums of $z_n$ can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms. Lemma 1 is similar to Apostol's chapter 1, problem 30. See [4] for a solution to this problem.

**Lemma 1.** *If $s_k^n = r/s$ with $r/s$ a reduced fraction, then $2^n$ divides $s$.*

*Proof.* The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, $a$; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, $na$. Also $k!$ will have a powers of 2 divisor with exponent $b$; and $(k!)^n$ will have a greatest power of 2 exponent of $nb$. Consider

$$\frac{(k!)^n}{(k!)^n}\sum_{j=2}^{k}\frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{1}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of $2^{nb-na}$ in their prime factorization, we have the numerator of (1) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and $A$ is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves $2^{na}$ as a factor in the denominator with no powers of 2 in the numerator, as needed. ☐

**Lemma 2.** *If $s_k^n = r/s$ with $r/s$ a reduced fraction and $p$ is a prime such that $k > p > k/2$, then $p^n$ divides $s$.*

*Proof.* First note that $(k, p) = 1$. If $p|k$ then there would have to exist $r$ such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^n}{(k!)^n}\sum_{j=2}^{k}\frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{2}$$

As $(k, p) = 1$, only the term $(k!)^n/p^n$ will not have $p$ in it. The sum of all such terms will not be divisible by $p$, otherwise $p$ would divide $(k!)^n/p^n$. As $p < k$, $p^n$ divides $(k!)^n$, the denominator of $r/s$, as needed. ☐

**Theorem 1.** *If $s_k^n = \frac{r}{s}$, with $r/s$ reduced, then $s > k^n$.*

*Proof.* Bertrand's postulate states that for any $k \geq 2$, there exists a prime $p$ such that $k < p < 2k$ [3]. For even $k$, we are assured that there exists a prime $p$ such that $k > p > k/2$. If $k$ is odd, $k-1$ is even and we are assured of the existence of prime $p$ such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies $k$ is even, a contradiction.

For both odd and even $k$, using Bertrand's postulate, we have assurance of the existence of a $p$ that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^n p^n$ divides the denominator of $r/s$ and as $2^n p^n > k^n$, the proof is completed. □

In light of this result we give the following definitions and corollary.

**Definition 1.**

$$D_{j^n} = \{0, 1/j^n, \ldots, (j^n-1)/j^n\} = \{0, .1, \ldots, .(j^n-1)\} \text{ \textit{base} } j^n$$

**Definition 2.**

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

**Corollary 1.**

$$s_k^n \notin \Xi_k^n$$

*Proof.* Reduced fractions are unique. Suppose, to obtain a contradiction, that there exists $a/b \in \Xi_k^n$ such that $a/b = r/s$ then $b < s$ by Theorem 1. If $a/b$ is not reduced, reduce it: $a/b = a_1/b_1$. A reduced fraction must have a smaller denominator than the unreduced form so $b_1 \leq b < s$ and this contradicts the uniqueness of the denominator of a reduced fraction. □

## 3 A Suggestive Table

The result of applying Corollary 1 to all partial sums of $z_2$ is given in Table 1.[1] The table shows that adding the numbers above each $D_{k^2}$, for all $k \geq 2$ gives results not in $D_{k^2}$ or any previous rows' such sets. So, for example, $1/4 + 1/9$ is not in $D_4$, $1/4 + 1/9$ is not in $D_4$ or $D_9$, $1/4 + 1/9 + 1/16$ is not in $D_4$, $D_9$, or $D_{16}$, etc.. That's what Corollary 1 says.

[1] Table 1 might remind readers of Cantor's diagonal method. We don't pursue this idea in this article.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $+1/4$ | | | | | | | |
| $+1/9$ | $+1/4$ | $+1/4$ | $+1/4$ | $+1/4$ | $\dots$ | $+1/4$ | |
| $\notin D_4$ | $+1/9$ | $+1/9$ | $+1/9$ | $+1/9$ | $\dots$ | $+1/9$ | |
| | $\notin D_9$ | $+1/16$ | $+1/16$ | $+1/16$ | | $\vdots$ | |
| | | $\notin D_{16}$ | $+1/25$ | $+1/25$ | | $\vdots$ | |
| | | | $\notin D_{25}$ | $+1/36$ | | $\vdots$ | |
| | | | | $\notin D_{36}$ | | | |
| | | | | | | $+1/(k-1)^2$ | |
| | | | | | | $+1/k^2$ | |
| | | | | | | $\notin D_{k^2}$ | |
| | | | | | | | $\ddots$ |

Table 1: A list of all rational numbers between 0 and 1 is given by the number sets along the diagonal. Partials of $z_2$ are excluded from sets below and to the upper left of the partial.

**Lemma 3.**

$$\lim_{k\to\infty} \Xi_k^n = \bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0,1)$$

*Proof.* Every rational $a/b \in (0,1)$ is included in at least one $D_{j^n}$. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in D_{b^n}$. □

Loosely speaking, Lemma 3 says that for all the series $z_n$ the denominators of their terms *cover* the possible rational convergence points and Corollary 1 says the partial sums of $z_n$ *escape* their terms.

# 4 Set theoretical proof

We will designate the set of rational numbers in $(0,1)$ with $\mathbb{Q}(0,1)$, the set of irrationals in $(0,1)$ with $\mathbb{H}(0,1)$, and the set of real numbers in $(0,1)$ with $\mathbb{R}(0,1)$. We use $\mathbb{R}(0,1) = \mathbb{Q}(0,1) \cup \mathbb{H}(0,1)$ and $\mathbb{Q}(0,1) \cap \mathbb{H}(0,1) = \emptyset$ in the following.

**Theorem 2.** *$z_n$ is irrational.*

*Proof.* Corollary 1 implies $s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n$. As $\lim_{k\to\infty} s_k^n = z_n$, using Lemma 3, we have

$$z_n \in \mathbb{R}(0,1) \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1).$$

That is $z_n$ is irrational. $\square$

# 5 Alternate proof

A proof by contradiction can also be given.

**Lemma 4.** *Given an $n$ there exists $x_m \in \Xi_k^m$, $m > n$, such that if $\epsilon = \min\{|x - s_k^n| : x \in \Xi_k^n\}$, then $|x_m - s_k^n| < \epsilon$ with $x_m \neq s_k^n$.*

*Proof.* As all values in $\Xi_k^n$ are repeated in $\Xi_k^m$ and as $\Xi_k^m$ contains, for some $m$, arbitrarily close rational approximations to all real (rational and irrational) numbers, the result follows. $\square$

**Theorem 3.** *$z_r$ is irrational.*

*Proof.* Suppose to obtain a contradiction that $z_r$ is rational. Then $z_r \in \Xi_k^r$ for some $k$ and hence for all $\Xi_l^n$ with $l > k$. But this violates the uniqueness of limits: a convergence point in $\Xi_k^r$ cannot also get arbitrarily close, per Lemma 4, to points in $\Xi_l^r$ that are different than points in $\Xi_k^r$; different points are guaranteed to exist per Lemma 3 and to be required per Corollary 1. $\square$

# 6 Conclusion

Both the set theoretical proof and the alternate proof point to the same very simple observational proof: rational approximations to an infinite series can't also be the series convergence point. As all rationals are given by the union of all decimal sets base $k^n$, no rational can be a convergence point. The series in question, $z_n$, is known to converge in $(0,1)$, so its convergence point must be irrational.

Using Table 1, one can understand this by placing $D_{\epsilon_1}$ after the first column's partial sum. $D_{\epsilon_1}$ designates the winning rational, in the sense that it gives the best approximation (within $\epsilon_1$) to the partial in the first column using any finite set of inexact rationals. Moving $D_{\epsilon_k}$ to successive columns the winning rational is updated to reflect the best approximation to all partial sums from say all decimal sets and partial sums encountered. The sequence $\epsilon_k$ converges to 0 just as all rationals are exhausted. This means the rational is forced to change, whereas in a series like $.\overline{1}$ in base 4, it stabilizes to $1/3$, the rational convergence point. The winning approximation is pulled back to $1/3$ versus propelled to the right perpetually. In a phrase: $z_n$ is perpetually offset from all rational values.

# References


[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

[2] F. Beukers, A Note on the Irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.*, **11**, (1979), 268–272.

[3] G. H. Hardy, E. M. Wright, R. Heath-Brown, J. Silverman, and A. Wiles, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, London, 2008.

[4] G. Hurst, Solutions to Introduction to Analytic Number Theory by Tom M. Apostol, Available at: https://greghurst.files.wordpress.com/2014/02/apostol_intro_to_ant.pdf