A Simple Proof that $\zeta(n \geq 2)$ is Irrational

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Abstract

We prove that partial sums of $\zeta(n) - 1 = z_n$ are not given by any single decimal in a number base given by a denominator of their terms. This result, applied to all partials, shows that partials are excluded from an ever greater number of rational, possible convergence points. The limit of the partials is z_n and the limit of the exclusions leaves only irrational numbers. Thus z_n is proven to be irrational.

1 Introduction

Beuker gives a proof that $\zeta(2)$ is irrational [2]. It is calculus based, but requires the prime number theorem, as well as subtle $\epsilon - \delta$ reasoning and generalizes only to the $\zeta(3)$ case. Here we give a simpler proof that uses just basic number theory [1] and treats all cases at once.

We use the following notation: for positive integers $n > 1$,

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

2 Decimals using denominators

Our aim in this section is to show that the reduced fractions that give the partial sums of z_n require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums of z_n can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms.

Lemma 1. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{1}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (1) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. □

Lemma 2. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{2}$$

As $(k, p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. □

Theorem 1. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$.*

Proof. Bertrand's postulate states that for any $k \geq 2$, there exists a prime p such that $k < p < 2k$ [3]. For even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Bertrand's postulate, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. $\square$

In light of this result we give the following definitions and corollary.

Definition 1.

$$D_{j^n} = \{0, 1/j^n, \ldots, (j^n-1)/j^n\} = \{0, .1, \ldots, .(j^n-1)\} \text{ } \textit{base } j^n$$

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

Proof. Reduced fractions are unique. Suppose, to obtain a contradiction, that there exists $a/b \in \Xi_k^n$ such that $a/b = r/s$ then $b < s$ by Theorem 1. If a/b is not reduced, reduce it: $a/b = a_1/b_1$. A reduced fraction must have a smaller denominator than the unreduced form so $b_1 \leq b < s$ and this contradicts the uniqueness of the denominator of a reduced fraction. $\square$

3 A Suggestive Table

The result of applying Corollary 1 to all partial sums of z_2 is given in Table 1.[1] The table shows that adding the numbers above each D_{k^2}, for all $k \geq 2$ gives results not in D_{k^2} or any previous rows' such sets. So, for example, $1/4 + 1/9$ is not in D_4, $1/4 + 1/9$ is not in D_4 or D_9, $1/4 + 1/9 + 1/16$ is not in D_4, D_9, or D_{16}, etc.. That's what Corollary 1 says.

[1] Table 1 might remind readers of Cantor's diagonal method. We don't pursue this idea in this article.

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\dots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\dots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 1: A list of all rational numbers between 0 and 1 is given by the number sets along the diagonal. Partials of z_2 are excluded from sets below and to the upper left of the partial.

Lemma 3.

$$\lim_{k\to\infty} \Xi_k^n = \bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0,1)$$

Proof. Every rational $a/b \in (0,1)$ is included in at least one D_{k^n}. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in D_{b^n}$. □

Loosely speaking, Lemma 3 says that for all the series z_n the denominators of their terms *cover* the possible rational convergence points and Corollary 1 says the partial sums of z_n *escape* their terms.

4 Set theoretical proof

We will designate the set of rational numbers in $(0,1)$ with $\mathbb{Q}(0,1)$, the set of irrationals in $(0,1)$ with $\mathbb{H}(0,1)$, and the set of real numbers in $(0,1)$ with $\mathbb{R}(0,1)$. We use $\mathbb{R}(0,1) = \mathbb{Q}(0,1) \cup \mathbb{H}(0,1)$ and $\mathbb{Q}(0,1) \cap \mathbb{H}(0,1) = \emptyset$ in the following.

Theorem 2. *z_n is irrational.*

Proof. Corollary 1 implies $s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n$. As $\lim_{k\to\infty} s_k^n = z_n$, using Lemma 3, we have

$$z_n \in \mathbb{R}(0,1) \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1).$$

That is z_n is irrational. $\square$

5 Conclusion

The set theoretical proof is a new trick; it is not in the accustomed $\epsilon - \delta$ format. Some reviewers have claimed that the geometric series refutes the logic of this proof. But placing the terms of a geometric series in a table, like Table 1, shows the problem: possible convergence points are not excluded. Others have claimed that a telescoping series like $1/2 - 1/3 + 1/3 - 1/4 + 1/4 - 1/5 + 1/5 - 1/6 + \cdots = 1/2$ refutes the proof given. In this case the terms cover the rationals, but the partials don't escape the series terms; the eight terms given add to 1/3. An effective counter-example must be a series that converges to a rational number with terms that *cover* all possible rational convergence points and has partials which *escape* these terms, when the denominators are used as number bases. Please re-read the above proof before attempting to construct a counter-example. Good luck!

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

[2] F. Beukers, A Note on the Irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.*, **11**, (1979), 268–272.

[3] G. H. Hardy, E. M. Wright, R. Heath-Brown, J. Silverman, and A. Wiles, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, London, 2008.