A Simple Proof that $\zeta(n \geq 2)$ is Irrational

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Abstract

We prove that partial sums of $\zeta(2) - 1 = z_2$ are not given by any single decimal in a number base given by a denominator of their terms. This result, applied to all partials, shows that partials are excluded from an ever greater number of rational values. The limit of the partials is z_2 and the limit of the exclusions leaves only irrational numbers.

1 Introduction

Beuker gives a proof that $\zeta(2)$ is irrational [3]. It is calculus based, but requires the prime number theorem, as well as subtle $\epsilon - \delta$ reasoning and generalizes only to the $\zeta(3)$ case. Here we give a simpler proof that uses just basic number theory [1] and does generalize to all other cases.

We use the following notation: for $n > 1$,

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n}.$$

2 Decimals using denominators

Our aim in this section is to show that the reduced fractions that give the partial sums of z_n require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums of z_n can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms.

We prove the general case.

Lemma 1. *The reduced fraction, r/s giving*

$$s_k^m = \sum_{j=2}^{k} \frac{1}{j^m} = \frac{r}{s} \tag{1}$$

is such that 2^m divides s.

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^m, 3^m, \ldots, k^m\}$ will have a greatest power of 2, ma. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^m$ will have a greatest power of 2 exponent of mb. Consider

$$\frac{(k!)^m}{(k!)^m} \sum_{j=2}^{k} \frac{1}{j^m} = \frac{(k!)^m/2^m + (k!)^m/3^m + \cdots + (k!)^m/k^m}{(k!)^m}. \tag{2}$$

The term $(k!)^m/2^{ma}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $mb - ma$ for 2. As all other terms but this term will have more than an exponent of 2^{mb-ma} in their prime factorization, we have the numerator of (2) has the form

$$2^{mb-ma}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^m/2^{ma}$. The denominator, meanwhile, has the factored form

$$2^{mb}C,$$

where $2 \nmid C$. This leaves 2^{ma} as a factor in the denominator with no powers of 2 in the numerator, as needed. □

Lemma 2. *If p is a prime such that $k > p > k/2$, then p^m divides s in (1).*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^m}{(k!)^m} \sum_{j=2}^{k} \frac{1}{j^m} = \frac{(k!)^m/2^m + \cdots + (k!)^m/p^m + \cdots + (k!)^m/k^m}{(k!)^m}. \tag{3}$$

As $(k,p)=1$, only the term $(k!)^m/p^m$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^m/p^m$. As $p<k$, p^m divides $(k!)^m$, the denominator of r/s, as needed. ∎

Theorem 1. *If*

$$s_k^m = \frac{1}{2^m} + \frac{1}{3^m} + \cdots + \frac{1}{k^m} = \frac{r}{s}, \tag{4}$$

with r/s reduced, then $s > k^m$.

Proof. Bertrand's postulate states that for any $k \geq 2$, there exists a prime p such that $k < p < 2k$ [4]. If k of (4) is even we are assured that there exists a prime p such that $k > p > k/2$. If k is odd $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Bertrand's postulate, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^m p^m$ divides the denominator of (4) and as $2^m p^m > k^m$, the proof is completed. ∎

In light of this result we give the following definitions and corollary for the z_2 case.

Definition 1.

$$D_{k^2} = \{0, 1/k^2, \ldots, (k^2-1)/k^2\} = \{0, .1, \ldots, .(k^2-1)\} \textit{ base } k^2$$

Definition 2.

$$\bigcup_{k=2}^{n} D_{k^2} = \Xi_n$$

Corollary 1.

$$s_n^2 \notin \Xi_n$$

Proof. This is an immediate consequence of Theorem 1. ∎

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\dots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\dots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 1: A list of all rational numbers between 0 and 1 is given by the number sets along the diagonal. Partials of z_2 are excluded from sets below and to the upper right of the partial.

3 A Suggestive Table

The result of applying Corollary 1 to all partial sums of z_2 is given in Table 1. The table shows that adding the numbers above each D_{k^2}, for all $k \geq 2$ gives results not in D_{k^2} or any previous rows' such sets. So, for example, $1/4+1/9$ is not in D_4, $1/4+1/9$ is not in D_4 or D_9, $1/4+1/9+1/16$ is not in D_4, D_9, or D_{16}, etc.. That's what Corollary 1 says. Note that every rational $a/b \in (0,1)$ is included in at least one D_{k^2}. For example, $ab/b^2 = a/b$, $a < b$ and so $a/b \in D_{b^2}$.

4 Set theoretical proof

We will designate the set of rational numbers in $(0,1)$ with $\mathbb{Q}(0,1)$, the set of irrationals in $(0,1)$ with $\mathbb{H}(0,1)$, and the set of real numbers in $(0,1)$ with $\mathbb{R}(0,1)$. We use $\mathbb{R}(0,1) = \mathbb{Q}(0,1) \cup \mathbb{H}(0,1)$ in the following.

Theorem 2. *z_2 is irrational.*

Proof. Theorem 1 implies

$$s_n^2 \in \mathbb{R}(0,1) \setminus \Xi_n.$$

As

$$\lim_{n\to\infty} s_n^2 = z_2$$

and

$$\lim_{n\to\infty} \Xi_n = \bigcup_{j=2}^{\infty} D_{j^2} = \mathbb{Q}(0,1),$$

$$z_n \in \mathbb{R}(0,1) \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1).$$

That is z_2 is irrational. □

5 Obviously wrong?!

A typical reaction to the above proof is that the geometric series shows that it is wrong. It can't be that elegant, simple, and correct. But the same treatment of the geometric series given by $.\overline{1}$, base 4 has the following parallel and supporting development:

$$g_n = \sum_{j=1}^{n} \frac{1}{4^j}$$

$$\lim_{n\to\infty} g_n = G = \frac{1}{3}$$

$$\Xi_{(4,n)} = \bigcup_{j=1}^{n} D_{4^j} = \{ \le n \text{ finite decimals base 4} \}$$

$$g_n \in \mathbb{R} \setminus \Xi_{(4,n-1)}$$

$$\lim_{n\to\infty} \Xi_{(4,n-1)} = \bigcup_{j=1}^{\infty} D_{4^j} = \{ \text{all finite decimals base 4} \} = \Xi_{(4,\infty)}$$

$$G \in \mathbb{R} \setminus \Xi_{(4,\infty)}.$$

This doesn't give a counter example to Theorem 2; it confirms its logic: 1/3 can't be expressed as a finite decimal in base 4.

6 Conclusion

This result for the irrationality of z_2 can be generalized; Theorem 1 gives a result for the general case; and all subsequent corollaries, tables, definitions, and lemmas can be easily modified for any $n > 2$.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

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[3] F. Beukers, A Note on the Irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.*, **11**, (1979), 268–272.

[4] G. H. Hardy, E. M. Wright, R. Heath-Brown, J. Silverman, and A. Wiles, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, London, 2008.