The Simplest Elementary Mathematics Proving Method of Fermat's Last Theorem

Haofeng Zhang
Beijing, China

Abstract: In this paper the author gives a simplest elementary mathematics method to solve the famous ***Fermat's Last Theorem*** (FLT), in which let this equation become a one unknown number equation, in order to solve this equation the author invented a method called "Order reducing method for equations" where the second order root compares to one order root and with some necessary techniques the author successfully proved ***Fermat's Last Theorem***.

1. Some Relevant Theorems

There are some theorems for proving or need to be known. ***All symbols in this paper represent positive integers unless they are stated to be not.***

Theorem 1.1. In the equation of

$$\begin{cases} x^n + y^n = z^n \\ \gcd(x, y, z) = 1 \\ n > 2 \end{cases} \tag{1-1}$$

x, y, z meet

$x \neq y$;

$x + y > z$;

if

$x > y$

then

$z > x > y$.

Proof: Let

$x = y$,

we have

$$2x^n = z^n$$

and

$$\sqrt[n]{2}x = z$$

where $\sqrt[n]{2}$ is not an integer and x, z are all positive integers, so $x \neq y$. Since

$$\left(x + y\right)^n = x^n + C_n^1 x^{n-1} y + ... + C_n^{n-1} x y^{n-1} + y^n > z^n,$$

so we get

$x + y > z.$

Since

$$x^n + y^n = z^n ,$$

so we have

$$z^n > x^n, z^n > y^n$$

and get

$$z > x > y$$

when

$$x > y .$$

Theorem 1.2. In the equation of (1-1), x, y, z meet

$$\gcd(x, y) = \gcd(y, z) = \gcd(x, z) = 1.$$

Proof: Since $x^n + y^n = z^n$, if $\gcd(x, y) > 1$ then we have $\left(x_1^{\ n} + y^n{}_1\right) \times [\gcd(x, y)]^n = z^n$ which causes $\gcd(x, y, z) > 1$ since the left side contains the factor of $[\gcd(x, y)]^n$ then the right side must also contains this factor but contradicts against (1-1) in which $\gcd(x, y, z) = 1$, so we have $\gcd(x, y) = 1$. Using the same way we have $\gcd(x, z) = \gcd(y, z) = 1$.

Theorem 1.3. In the equation of (1-1), x, y, z meet

$$x^{n-i} + y^{n-i} > z^{n-i} ,$$

$$x^{n+i} + y^{n+i} > z^{n+i} ,$$

where

$$n > i \geq 1.$$

Proof: From equation (1-1), since

$$x^n + y^n = z^n ,$$

from **Theorem 1.1**, since $z > x > y$, we have

$$x^{n-i} + y^{n-i} > \left[\left(\frac{x}{z}\right)^i x^{n-i} + \left(\frac{y}{z}\right)^i y^{n-i} = z^{n-i} \right],$$

$$x^{n+i} + y^{n+i} < \left(z^i x^{n-i} + z^i y^{n-i} = z^{n+i}\right),$$

so we have

$$x^{n-i} + y^{n-i} > z^{n-i} .$$

$$x^{n+i} + y^{n+i} < z^{n+i} .$$

This theorem means given x, y, z if equation (1-1) has one positive integer solution then this solution is the only one.

Theorem 1.4. There are no positive integer solutions for

$$x^3 + y^3 = z^3.$$

Proof:

For the 3 orders equation $ax^3 + bx^2 + cx + d = 0$, the real root expression is

$$\begin{cases} p = \dfrac{c}{a} - \dfrac{b^2}{3a^2} \\ q = \dfrac{d}{a} + \dfrac{2b^3}{27a^3} - \dfrac{bc}{3a^2} \\ Y = \sqrt[3]{-\dfrac{q}{2} + \sqrt{\left(\dfrac{q}{2}\right)^2 + \left(\dfrac{p}{3}\right)^3}} + \sqrt[3]{-\dfrac{q}{2} - \sqrt{\left(\dfrac{q}{2}\right)^2 + \left(\dfrac{p}{3}\right)^3}} \\ x = Y - \dfrac{b}{3a} \end{cases}.$$

Let

$$\begin{cases} x > y \\ y = x - f \\ z = x + e. \end{cases}$$

We get

$$x^n + (x - f)^n = (x + e)^n.$$

When n=3 we have:

$$x^3 + (x - f)^3 = (x + e)^3.$$

After sorting we get

$$x^3 - 3(f + e)x^2 + 3(f^2 - e^2)x - (f^3 + e^3) = 0$$

So

$$\begin{cases} p = -6e(e + f) \\ q = -3e(2e^2 + 3ef + f^2) \\ \sqrt{\left(\dfrac{q}{2}\right)^2 + \left(\dfrac{p}{3}\right)^3} = \sqrt{\left(\dfrac{-3e(2e^2 + 3ef + f^2)}{2}\right)^2 + \left(\dfrac{-6e(e + f)}{3}\right)^3} = k_0 \\ e^2(e + f)^2(4e^2 + 4ef + 9f^2) = 4k_0^{\ 2} \end{cases}.$$

We have

$$x=(e+f)+\sqrt[3]{\frac{e(e+f)\left(3(2e+f)+2\sqrt{4e^2+4ef+9f^2}\right)}{2}}+\sqrt[3]{\frac{e(e+f)\left(3(2e+f)-2\sqrt{4e^2+4ef+9f^2}\right)}{2}}$$

If

$$\begin{cases}\sqrt[3]{\dfrac{3(2e+f)+2\sqrt{4e^2+4ef+9f^2}}{2}}\neq K_1\sqrt[3]{e^2(e+f)^2}\\ \sqrt[3]{\dfrac{3(2e+f)-2\sqrt{4e^2+4ef+9f^2}}{2}}\neq K_2\sqrt[3]{e^2(e+f)^2}\end{cases}\tag{1-2}$$

then

$$\sqrt[3]{\frac{e(e+f)\left(3(2e+f)+2\sqrt{4e^2+4ef+9f^2}\right)}{2}}+\sqrt[3]{\frac{e(e+f)\left(3(2e+f)-2\sqrt{4e^2+4ef+9f^2}\right)}{2}}$$
$$=K_3$$

and

$$6e(e+f)(2e+f)+$$
$$3e(e+f)\sqrt[3]{20e^2+20ef-27f^2}\left[\begin{array}{l}\sqrt[3]{3(2e+f)+2\sqrt{4e^2+4ef+9f^2}}+\\ \sqrt[3]{3(2e+f)-2\sqrt{4e^2+4ef+9f^2}}\end{array}\right]$$
$$=2K_3{}^3$$

So

$$\sqrt[3]{3(2e+f)+2\sqrt{4e^2+4ef+9f^2}}+\sqrt[3]{3(2e+f)-2\sqrt{4e^2+4ef+9f^2}}$$
$$=\frac{2K_3{}^3-6e(e+f)(2e+f)}{3e(e+f)\sqrt[3]{20e^2+20ef-27f^2}}$$

and

$$6e(e+f)(2e+f)+$$
$$3e(e+f)\sqrt[3]{20e^2+20ef-27f^2}\left[\begin{array}{l}\sqrt[3]{3(2e+f)+2\sqrt{4e^2+4ef+9f^2}}+\\ \sqrt[3]{3(2e+f)-2\sqrt{4e^2+4ef+9f^2}}\end{array}\right]$$
$$=\left[\frac{2K_3{}^3-6e(e+f)(2e+f)}{3e(e+f)}\right]^3\frac{1}{20e^2+20ef-27f^2}$$

Repeat this step we can do it forever, the value of

$$6e(2e+f)+3e(e+f)\sqrt[3]{20e^2+20ef-27f^2}\left[\begin{array}{l}\sqrt[3]{3(2e+f)+2\sqrt{4e^2+4ef+9f^2}}\ +\\ \sqrt[3]{3(2e+f)-2\sqrt{4e^2+4ef+9f^2}}\end{array}\right]$$

is decreasing, but the positive integer value can not be smaller than 1 that means only finite repeating steps can we do and at last we must find that only non integers can exist which means there are no positive integer solutions at the situation of (1-2) . So we have

$$\begin{cases}\sqrt[3]{\dfrac{3(2e+f)+2\sqrt{4e^2+4ef+9f^2}}{2}}=K_1\sqrt[3]{e^2(e+f)^2}\\ \sqrt[3]{\dfrac{3(2e+f)-2\sqrt{4e^2+4ef+9f^2}}{2}}=K_2\sqrt[3]{e^2(e+f)^2}\end{cases}.$$

And

$$\begin{cases}\dfrac{3(2e+f)+2\sqrt{4e^2+4ef+9f^2}}{2}=K_1^{\ 3}e^2(e+f)^2\\ \dfrac{3(2e+f)-2\sqrt{4e^2+4ef+9f^2}}{2}=K_2^{\ 3}e^2(e+f)^2\end{cases}.$$

So we get

$$\begin{cases}3(2e+f)=\left(K_1^{\ 3}+K_2^{\ 3}\right)e^2(e+f)^2\\ 2\sqrt{4e^2+4ef+9f^2}=\left(K_1^{\ 3}-K_2^{\ 3}\right)e^2(e+f)^2\end{cases}$$

but impossible because if not we get

$$\begin{cases}\dfrac{9(2e+f)^2}{(e+f)^4}=\left(K_1^{\ 3}+K_2^{\ 3}\right)^2\\ \dfrac{4\left(4e^2+4ef+9f^2\right)}{(e+f)^4}=\left(K_1^{\ 3}-K_2^{\ 3}\right)^2\end{cases}.$$

And we have

$$\frac{9(2e+f)^2}{(e+f)^4}-\frac{4\left(4e^2+4ef+9f^2\right)}{(e+f)^4}=\left(K_1^{\ 3}+K_2^{\ 3}\right)^2-\left(K_1^{\ 3}-K_2^{\ 3}\right)^2=4K_1^{\ 3}K_2^{\ 3}$$

in which

$$20e^2+20ef-27f^2=K_1^{\ 3}K_2^{\ 3}\left(4e^4+16e^3f+24e^2f^2+16ef^3+4f^4\right)$$

that is impossible since the absolute value of left side is smaller than the absolute value of right side. If $K_2=0$ then we have

$$3(2e+f)-2\sqrt{4e^2+4ef+9f^2}=0$$

in which

$$20e^2 + 20ef - 27f^2 = 0$$

and

$$e = \left(\frac{-20 + \sqrt{2560}}{40} \right) f$$

that means e is not an integer. So when $n = 3$ there are no positive integer solutions for equation (1-1).

Theorem 1.5. There are no positive integer solutions for

$$1^n + y^n = z^n .$$

Proof: Since

$$1 = z^n - y^n = (z - y)(z^{n-1} + z^{n-2}y + \dots + zy^{n-2} + y^{n-1})$$

where

$$\begin{cases} z - y = 1 \\ (z^{n-1} + z^{n-2}y + \dots + zy^{n-2} + y^{n-1}) = 1 \end{cases}$$

that causes z, y to be non positive integers, so there are no positive integer solutions for

$$1^n + y^n = z^n .$$

Theorem 1.6. There are no positive integer solutions for

$$2^n + y^n = z^n .$$

Proof: Since

$$2^n = z^n - y^n = (z - y)(z^{n-1} + z^{n-2}y + \dots + zy^{n-2} + y^{n-1}),$$

if

$$\begin{cases} z - y = 1 \\ z^{n-1} + z^{n-2}y + \dots + zy^{n-2} + y^{n-1} = 2^n \end{cases}$$

then taking the least value for $y = 2, z = 3$, we have

$$3^{n-1} + 2 \times 3^{n-2} + \dots + 2^{n-1} > 2^n$$

when $n > 2$ that is impossible. If

$$\begin{cases} z - y = 2^i \\ z^{n-1} + z^{n-2}y + \dots + zy^{n-2} + y^{n-1} = 2^j \\ i + j = n \\ i \ge 1 \end{cases}$$

then $z > 2$ and taking the least value of $y = 2, z = 3$, we get

$$3^{n-1} + 2 \times 3^{n-2} + \ldots + 2^{n-1} > 2^j$$

with $n > 2$ that is also impossible, so there are no positive integer solutions for

$$2^n + y^n = z^n .$$

Theorem 1.7. There are no positive integer solutions for equation (1-1) when $n \to \infty$ and x, y, z in equation (1-1) meet

$$\sqrt[n]{2}y < z < \sqrt[n]{2}x,$$
$$x > 2,$$
$$y > 1,$$
$$z > 3.$$

Proof: Since $x^n + y^n = z^n$, let $x > y$, we get

$$\left(\frac{z}{x}\right)^n - \left(\frac{y}{x}\right)^n = 1,$$

since

$$z > x > y,$$

so we have

$$z < \sqrt[n]{2}x,$$

and

$$\lim_{n\to\infty}\left(\frac{z}{x}\right)^n - \left(\frac{y}{x}\right)^n = \infty > 1$$

which means there are no positive integer solutions for equation (1-1) when $n \to \infty$. And according to **Theorem 1.1, 1.6** we have $x > 2, y > 1, z > 3$.

If $y^n \geq \frac{z^n}{2}$ then since $x > y$, so we have

$$x^n + y^n > z^n,$$

that is impossible so we have

$$\sqrt[n]{2}y < z .$$

Theorem 1.8. There are no positive integer solutions for equation (1-1) when $x, y, z \leq 10$.

Proof: There are below combinations of x, y, z when $x, y, z \leq 10$:

$3^n + 4^n = 5^n$,

$3^n + 5^n = 6^n$,

$3^n + 6^n = 7^n$,

$3^n + 7^n = 8^n$,

$3^n + 8^n = 9^n$,

$3^n + 9^n = 10^n$,

$4^n + 5^n = 6^n$,

$4^n + 6^n = 7^n$,

$4^n + 7^n = 8^n$,

$4^n + 8^n = 9^n$,

$4^n + 9^n = 10^n$,

$5^n + 6^n = 7^n$,

$5^n + 7^n = 8^n$,

$5^n + 8^n = 9^n$,

$5^n + 9^n = 10^n$,

$6^n + 7^n = 8^n$,

$6^n + 8^n = 9^n$,

$6^n + 9^n = 10^n$,

$7^n + 8^n = 9^n$,

$7^n + 9^n = 10^n$,

$8^n + 9^n = 10^n$.

Here we take $7^n + 9^n = 10^n$ for example to explain how to prove. We plot the graph for this equation as showed in **Figure 1-1**.

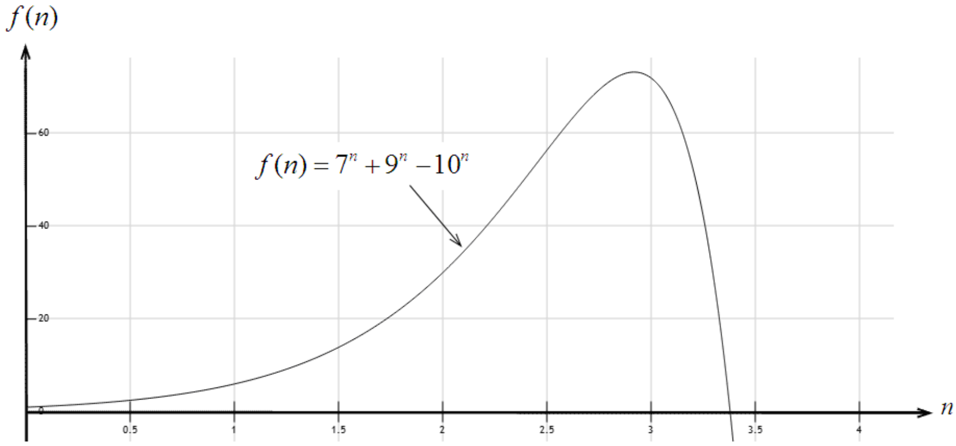

Figure 1-1 *Graph of* $f(n) = 7^n + 9^n - 10^n$

Obviously for equation $f(n) = 7^n + 9^n - 10^n$ in **Figure 1-1**, we have $3 < n < 4$ is not an integer so there are no positive integer solutions for it, and so it is with other cases and we have the conclusion of there are positive integer solutions for equation (1-1) when $x, y, z \le 10$.

2. Proving Method

In equation (1-1), let

$$\begin{cases} a = x^{n-2} \\ b = y^{n-2} \\ c = z^{n-2} \end{cases},$$

we have

$$\begin{cases} ax^2 + by^2 = cz^2 \\ a^{\frac{n-1}{n-2}} x + b^{\frac{n-1}{n-2}} y = c^{\frac{n-1}{n-2}} z \end{cases}. \tag{2-1}$$

Since we reduce the order of equation so the method is called "Order reducing method for equations".

Let $x > y$ and

$$\begin{cases} y = x - f \\ z = x + e \end{cases}. \tag{2-2}$$

From (2-1) and (2-2) we have

$$\begin{cases} ax^2 + b(x-f)^2 = c(x+e)^2 \\ a^{\frac{n-1}{n-2}}x + b^{\frac{n-1}{n-2}}(x-f) = c^{\frac{n-1}{n-2}}(x+e) \end{cases}$$

and

$$\begin{cases} (a+b-c)x^2 - 2(bf+ce)x + \left(bf^2 - ce^2\right) = 0 \\ a^{\frac{n-1}{n-2}}x + b^{\frac{n-1}{n-2}}(x-f) - c^{\frac{n-1}{n-2}}(x+e) = 0 \end{cases},$$

the roots are

$$x = \frac{(bf+ce) \pm \sqrt{(bf+ce)^2 - (a+b-c)\left(bf^2 - ce^2\right)}}{x^{n-2} + y^{n-2} - z^{n-2}}, \tag{2-3}$$

and

$$x = \frac{c^{\frac{n-1}{n-2}}e + b^{\frac{n-1}{n-2}}f}{a^{\frac{n-1}{n-2}} + b^{\frac{n-1}{n-2}} - c^{\frac{n-1}{n-2}}} = \frac{bfy + cez}{x^{n-1} + y^{n-1} - z^{n-1}}. \tag{2-4}$$

There are two cases for bf^2, ce^2 when $bf^2 \geq ce^2$ and $bf^2 < ce^2$. There are also two cases for $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}}$ when $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} \leq 1$ and $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} > 1$.

Case A: If $bf^2 \geq ce^2$, from (2-3) when

$$x = \frac{(bf+ce) + \sqrt{(bf+ce)^2 - (a+b-c)\left(bf^2 - ce^2\right)}}{x^{n-2} + y^{n-2} - z^{n-2}},$$

From **Theorem 1.3** we know $a+b-c = x^{n-2} + y^{n-2} - z^{n-2} > 0$, so we have

$$x \leq \frac{2(bf+ce)}{x^{n-2} + y^{n-2} - z^{n-2}},$$

and also from **Theorem 1.3** we have $x^{n-1} + y^{n-1} - z^{n-1} > 0$, compare to (2-4) we get

$$\frac{bfy + cez}{x^{n-1} + y^{n-1} - z^{n-1}} \leq \frac{2(bf+ce)}{x^{n-2} + y^{n-2} - z^{n-2}}.$$

When $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} \leq 1$, we have

$$bfy + cez \leq 2(bf+ce),$$

that is impossible since from **Theorem 1.7** we know $y \geq 2$ and $z > 3$. So in order to have positive integer solutions, $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}$ must satisfy $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}} > 1$. But from **Theorem 1.4** we know there are no positive integer solutions for equation (1-1) when $n = 3$, no matter how many times we increase $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}$. So there are still no positive integer solutions when $n > 3$. In other words to say is that since n is for all the positive integers bigger than 2, so if one of them does not have positive integer solutions then other numbers still do not have. For example, if $n = 4$ there have positive integer solutions for equation (1-1) then there must also have positive integer solutions when $n = 3$.

When

$$x = \frac{(bf+ce) - \sqrt{(bf+ce)^2 - (a+b-c)(bf^2 - ce^2)}}{x^{n-2}+y^{n-2}-z^{n-2}}.$$

we have

$$x \leq \frac{bf+ce}{x^{n-2}+y^{n-2}-z^{n-2}},$$

compare to (2-4) we get

$$\frac{bfy+cez}{x^{n-1}+y^{n-1}-z^{n-1}} \leq \frac{bf+ce}{x^{n-2}+y^{n-2}-z^{n-2}}.$$

When $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}} \leq 1$, we have

$$bfy + cez \leq bf + ce,$$

that is impossible since from **Theorem 1.7** we know $y \geq 2$ and $z > 3$. So in order to have positive integer solutions, $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}$ must satisfy $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}} > 1$. But from **Theorem 1.4** we know there are no positive integer solutions for equation (1-1) when $n = 3$, even though $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}} > 1$, so there still no positive integer solutions when $n > 3$.

Case B: If $bf^2 < ce^2$, from (2-3) when

$$x = \frac{(bf + ce) + \sqrt{(bf + ce)^2 + (a + b - c)(ce^2 - bf^2)}}{x^{n-2} + y^{n-2} - z^{n-2}},$$

we can prove $(bf + ce)^2 > (a + b - c)(ce^2 - bf^2)$ since if not we have

$$(bf + ce)^2 \leq (a + b - c)(ce^2 - bf^2)$$

and

$$[(2b + a) - c]bf^2 + 2bfce + [2c - (a + b)]ce^2 \leq 0$$

that is impossible since $a + b - c > 0$ and $c > a, c > b, 2c - (a + b) > 0.$ So we have

$$x < \frac{(bf + ce)(1 + \sqrt{2})}{x^{n-2} + y^{n-2} - z^{n-2}}$$

compare to (2-4) we get

$$\frac{bfy + cez}{x^{n-1} + y^{n-1} - z^{n-1}} < \frac{(bf + ce)(1 + \sqrt{2})}{x^{n-2} + y^{n-2} - z^{n-2}}.$$

When $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} \leq 1$, we have

$$bfy + cez < (bf + ce)(1 + \sqrt{2}) < 2.5(bf + ce)$$

and

$$bf(x - f) + ce(x + e) < 2.5(bf + ce)$$

that leads to

$$x < \left[\frac{2.5(bf + ce) + bf^2 - ce^2}{bf + ce} = 2.5 - \frac{ce^2 - bf^2}{bf + ce}\right] < 2.5$$

where possible values for x are 1, 2 but according to **Theorem 1.6**, **1.7** we know there are no positive integer solutions. So in order to have positive integer solutions, $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}}$ must satisfy $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} > 1$. But from **Theorem 1.4** we know there are no positive integer solutions for equation (1-1) when $n = 3$, even though $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} > 1$, so there still no positive integer solutions when $n > 3$.

When

$$x = \frac{(bf + ce) - \sqrt{(bf + ce)^2 + (a + b - c)(ce^2 - bf^2)}}{x^{n-2} + y^{n-2} - z^{n-2}}$$

is not possible since $x \le 0$.

Now we have completely solved no positive integer solutions for equation (1-1) when $n > 2$ using "Order reducing method for equations".

3. Conclusion

Through the above contents we can see clearly that the proving of *Fermat's Last Theorem* is just a problem of elementary mathematics. "Order reducing method for equations" that the author invented is a very effective method in the proving of *Fermat's Last Theorem* and the author's technique in which let $y = x - f$ and $z = x + e$ is a very important step for solving.

Fermat's Last Theorem is a problem that has lasted for about 380 years. Proving methods are not important but the theorem's correctness is very necessary because many useful inferences can be deduced that are obviously better than "conjectures".

The author has been working on proving of *Fermat's Last Theorem* for quite some times (251 days) without any reference and many methods have been thought about, for example "Method of prime factorization" but not work. So the author has already known that there are no ways to solve except "Solving high order equations" which is also an important aspect in solving other mathematic problems.

Using the method in this paper we can prove there are still no positive integer solutions for equation

$$x_1^{\,n} + x_2^{\,n} + \ldots + x_i^{\,n} = x_{i+1}^{\;n},$$

when $n > p$, where p is a prime number under the assumption of no positive integer solutions for $x_1^{\,p} + x_2^{\,p} + \ldots + x_i^{\,p} = x_{i+1}^{\;p}$. For example $x_1^{\,n} + x_2^{\,n} + x_3^{\,n} = x_4^{\,n}$, if we can prove there are no positive integer solutions for $x_1^{\,5} + x_2^{\,5} + x_3^{\,5} = x_4^{\,5}$ or $x_1^{\,7} + x_2^{\,7} + x_3^{\,7} = x_4^{\,7}$ then there are still no positive integer solution for $x_1^{\,n} + x_2^{\,n} + x_3^{\,n} = x_4^{\,n}$ when $n > 5$ or $n > 7$.