A Lemma on the Minimal Counter-example of Frankl's Conjecture

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Abstract

Frankl's Conjecture, from 1979, states that any finite union-closed family, containing at least one non-empty member set, must have an element which belongs to at least half of the member-sets. In this paper we list out some properties of the hypothetical minimal counter-example to this conjecture. In particular, we discuss the frequency of 3 distinct elements in the minimal counter-example. We also apply these findings to finite bipartite graphs.

1 Introduction

A family of sets $\mathcal{A}$ is said to be union-closed if the union of any two member sets is also a member of $\mathcal{A}$. Peter Frankl's conjecture (or the union-closed sets conjecture) states that if $\mathcal{A}$ is finite, then some element must belong to at least half of the sets in $\mathcal{A}$, provided at least one member set is non-empty. A detailed discussion and current standing of the conjecture can be found in [1].

One approach to the proof of this conjecture is to directly attack its minimal counter-example as in [3]. The main result (Lemma 3) that we prove here in this direction is the following: if the cardinality of minimal counter-example family be $2n+1$, then there must exist at least 3 distinct elements belonging to **exactly** n sets. We also construct an augmentation technique and attempt to classify the union-closed families based on their cardinalities. Finally we translate the main result to the field of graph theory (Lemma 5).

2 Main results

We call a family $\mathcal{A}$ as *not Frankl's* if it violates the union-closed sets conjecture.

We define the *frequency* of an element x as the number of member sets in $\mathcal{A}$ that x belongs to.

2.1 An augmentation technique

In this section we develop a technique to augment a union-closed family by exactly one set, while preserving the union-closed property. The significance of this method is that the augmented family contains only one set different from the original family.

Lemma 1: Any union closed family $\mathcal{A}$ with $|\mathcal{A}| = n$ can be augmented to generate a new union-closed family $\mathcal{A}'$ with $|\mathcal{A}'| = n + 1$.

Proof: We inject a new set X in the family $\mathcal{A}$. This new set may be constructed as following:

$X = \{\{\cup\mathcal{A}\} \cup \{x\} : x \notin \{\cup\mathcal{A}\}\}$

The frequencies of each element in this new family $\mathcal{A}' = \mathcal{A} \cup \{X\}$ gets increased by 1.

We use the above technique to attempt the classification of union-closed families based on their cardinalities.

Definition 1: We construct a set $C \subset \mathbb{N}$ as $C := \{y \in \mathbb{N} : \exists\ \mathcal{A}$ with $|\mathcal{A}| = y$ and $\mathcal{A}$ is not Frankl's$\}$. We set $E := \mathbb{N} \setminus C$.

Theorem 1: For $n \in \mathbb{N}$, if $2n \in C$, then $2n + 1 \in C$.

Proof: We prove this by contradiction. As $2n \in C$, $\exists$ a union-closed $\mathcal{A}$ with $|\mathcal{A}| = 2n$ and is not Frankl's. The maximum frequency of any element can therefore be $n - 1$. We augment $\mathcal{A}$ to $\mathcal{A}'$ using the method in *Lemma 1.* As $|\mathcal{A}'| = 2n + 1$ and frequency of any element in $\mathcal{A}'$ cannot be more than $(n - 1) + 1 = n$, hence $\mathcal{A}'$ is not Frankl's. Therefore $2n + 1 \in C$.

2.2 Properties of the minimal counter-example

We now explore the properties of the minimal counter-example. We formally define the minimal counter-example as follows:

Definition 2: We call a union-closed family $\mathcal{A}$ a *minimal counter-example* if it is not Frankl's and $|\mathcal{A}| = \inf C$

We define the basis sets of a union-closed family:

Definition 3: A set $B \in \mathcal{A}$ ($\mathcal{A}$ being union-closed) is called a *basis* if there are *no* sets X,$Y \in \mathcal{A}$ [$X \neq Y$ and, X or $Y \neq \emptyset$] such that $X \cup Y = B$.

Note: We observe that removing a basis set from a union-closed family generates a new family which is also union-closed.

To prove our main lemma, we must safely remove 3 basis sets from a union-closed $\mathcal{A}$. Hence, we prove the following lemma first:

Lemma 2: If $\mathcal{A}$ is a minimal counter-example, then it must contain at least 6 basis sets.

Proof: From [4], we know that the minimal counter-example $\mathcal{A}$ must contain 51 sets. Let b be the number of basis sets of $\mathcal{A}$. All the sets of $\mathcal{A}$ are formed by the union of 2 or more basis sets, hence the maximum cardinality of $\mathcal{A}$ can be $\binom{b}{1} + \binom{b}{2} + ... + \binom{b}{b} = 2^b$. But, $|\mathcal{A}| = 51$. Therefore, $2^b \geq 51$, or $b > 5$.

Theorem 2: The integer inf C must be odd.

Proof: We prove this using contradiction. Let $\inf C = 2n$ (note that $\inf C \in C$), n being a natural number. Then there exists a family $\mathcal{A}$ with $|\mathcal{A}| = 2n$ and is not Frankl's. We derive a new union-closed family $\mathcal{A}'$ by removing a basis set from $\mathcal{A}$. As $|\mathcal{A}'| = 2n - 1 < 2n$, so $|\mathcal{A}'| \in E$. Thus, $\mathcal{A}'$ must be Frankl's. Therefore, $\mathcal{A}'$ must contain an element with frequency n. As, the same element has frequency n in $\mathcal{A}$, hence $\mathcal{A}$ must also be Frankl's. Therefore, we reach a contradiction.

We now proceed to the main lemma of this article.

Lemma 3: If $\mathcal{A}$ be a minimal counter-example with $|\mathcal{A}| = 2n + 1$, then $\mathcal{A}$ must contain at least 3 distinct elements with frequency *exactly equal* to n.

Proof: By *Lemma 2*, $\mathcal{A}$ must contain at least 6 basis sets. If we remove a basis set B, we get a new union-closed family $\mathcal{A}_B$ with $|\mathcal{A}_B| = 2n$. As $2n \in E$, there must exist an element x_1 in at least n sets of $\mathcal{A}_B$. The frequency of x_1 cannot be greater than n as it will render $\mathcal{A}$ Frankl's (a contradiction). Hence, frequency of x_1 must be n.

Now, as x_1 must belong to at least one basis set B_1, we remove B_1 from $\mathcal{A}$ to get $\mathcal{A}_{B_1}$ with $|\mathcal{A}_{B_1}| = 2n$. Again $\mathcal{A}_{B_1}$ must be Frankl's with x_1 occurring in $n - 1$ sets. Hence, there must exist another element x_2 present in exactly n sets of $\mathcal{A}_{B_1}$. Also, $x_2 \notin B_1$, otherwise frequency of x_2 in $\mathcal{A}$ will become $n + 1$ making it Frankl's (a contradiction). Let $x_2 \in B_2$ (a basis) where $B_1 \neq B_2$.

Next, we remove both B_1 and B_2 from $\mathcal{A}$ to get a new family $\mathcal{A}_{B_1 B_2}$. The frequency of x_1 in $\mathcal{A}_{B_1 B_2}$ becomes $n-1$ [when $x_1 \notin B_2$] or $n-2$ [when $x_1 \in B_2$]. x_2 has a frequency of $n-1$ in $\mathcal{A}_{B_1 B_2}$. As $|\mathcal{A}_{B_1 B_2}| = 2n-1$, it is Frankl's. Hence, we must have another element x_3, distinct from x_1 and x_2, with a frequency n in $\mathcal{A}_{B_1 B_2}$. x_3 cannot be an element of B_1 or B_2, else $\mathcal{A}$ would be Frankl's.

Therefore, we have at least 3 distinct elements, namely x_1, x_2 and x_3 with an exact frequency of n.

2.3 Translation to Graph Theory

It is well known that the union-closed sets conjecture is equivalent to the intersection-closed sets conjecture. We state the intersection-closed sets conjecture here:

Conjecture 1: For any intersection-closed family of sets that contains more than one set, there exists an element that belongs to at most half of the sets in the family.

We do not repeat the proof of equivalence of the two conjectures here, but highlight the key ideas, which will help us to smoothly transfer our results to the field of graph theory.

For every union-closed $\mathcal{A}$ we can construct $\mathcal{A}'$, with $|\mathcal{A}'| = |\mathcal{A}|$, where, if $A \in \mathcal{A}$ then the complement of A, i.e. $\{\cup\mathcal{A}\} \setminus A \in \mathcal{A}'$. It is easy to see that this family of complement sets $\mathcal{A}'$ is intersection-closed. Next, for any element x we define the frequency of x in $\mathcal{A}$ as $|\mathcal{A}_x|$. The number of sets which do not contain x is defined as $|\mathcal{A}_{\overline{x}}|$.

Therefore, for a union-closed $\mathcal{A}$ and its corresponding intersection-closed $\mathcal{A}'$, we have $|\mathcal{A}'_x| = |\mathcal{A}_{\overline{x}}|$ for any element x in $\mathcal{A}'$. Also, the fact that $|\mathcal{A}_{\overline{x}}| = |\mathcal{A}|$ - $|\mathcal{A}_x|$ is trivial. With these ideas we are in a position to state *Lemma 3* for intersection-closed minimal counter-example.

Lemma 4: If the minimal counter-example $\mathcal{A}'$ to the intersection-closed conjecture contain $2n + 1$ sets, then there must be 3 distinct elements in $\mathcal{A}'$ contained in exactly $n + 1$ sets.

Proof: Let $\mathcal{A}$ be the union-closed family containing the complement sets of $\mathcal{A}'$, then from *Lemma 3* we have 3 distinct elements x_1, x_2 and x_3 contained in exactly n sets of $\mathcal{A}$. As $|\mathcal{A}'_{x_1}| = |\mathcal{A}_{\overline{x_1}}|$, therefore $|\mathcal{A}'_{x_1}| = |\mathcal{A}|$ - $|\mathcal{A}_{x_1}| = (2n+1)-n$ $= n + 1$. Similarly, $|\mathcal{A}_{x_2}| = n + 1$ and $|\mathcal{A}_{x_3}| = n + 1$.

The graph theory equivalent of intersection-closed sets conjecture is described in [2]:

Conjecture 2: Let G be a finite bipartite graph with at least one edge. Then each of the two bipartition classes contains a vertex belonging to at most half of the maximal stable sets.

We can then translate the findings of *Lemma 4* to the context of graph theory to prove the following lemma:

Lemma 5: Let the minimal counter-example bipartite graph G to *Conjecture 2* contain $2n + 1$ maximal stable sets. Then each bipartition class, X, Y of G, must contain 3 distinct vertices present in exactly $n + 1$ maximal stable sets of G.

Proof: Let $\mathcal{S}$ be the family of maximal stable sets of G. It is shown in [2], that the family $\mathcal{X} := \{X \cap S\colon\ S \in \mathcal{S}\}$ is intersection-closed. Same holds true for the family $\mathcal{Y} := \{Y \cap S\colon\ S \in \mathcal{S}\}$. Thus, from *Lemma 4* we have 3 vertices in each partition X and Y occurring in exactly $n+1$ sets of $\mathcal{X}$ and $\mathcal{Y}$ respectively.

3 Remarks and a Hypothesis

The result proved in *Theorem 1* is a weak one. Ideally we would like to prove the stronger result, sup $E + 1 =$ inf C. *Theorem 1* only proves one half of this. The other half would be to show that if $2n - 1 \in C$, then $2n \in C$. We need a better augmentation technique than that in *Lemma 1* to prove the other half.

In the article [3], the author has defined the set C_a as $\cup\mathcal{A}_{\overline{a}}$. The family containing all such C_a is $\mathcal{C}$. If $|\cup\mathcal{A}| = q$, then $|\mathcal{C}| = q-1$. Also, we have $|\mathcal{A}_{\overline{x_1}}|$ $= |\mathcal{A}_{\overline{x_2}}| = |\mathcal{A}_{\overline{x_3}}| = n+1$. Therefore, there exists an element p in at least half the sets of $\mathcal{A}_{\overline{x_1}}$, as $\mathcal{A}_{\overline{x_1}}$ is Frankl's. Again, from [3], we have that $\mathcal{C} \setminus C_{x_1} \subset \mathcal{A}_{x_1}$, $\mathcal{C} \setminus C_p \subset \mathcal{A}_p$ and $\{\cup\mathcal{A}\} \in \mathcal{A}_p$. Therefore $|\mathcal{A}_{x_1} \cap \mathcal{A}_p| \geq q-1$. Now, if we can show that there exist k sets each containing p, but is not a member of $\mathcal{C}$ $\cup\ \mathcal{A}_{\overline{x_1}} \cup \{\cup\mathcal{A}\}$, then we have the inequality $((n+1)/2)+q-1+k \leq n$, which yields $|\mathcal{A}| \geq 4q-1+4k$. We intuit that $k > 0$, but a better estimate of k eludes us. Currently, we know $k \geq 0$, but the results in this paper may help to find a better lower bound for k.

References

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