

POSITIVITY OF LI COEFFICIENTS FOR $n > 10^{24}$

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ABSTRACT. We investigate the Riemann Zeta function, in the form $(s-1)\zeta(s)$, under the transformation $s = \frac{1}{1-z}$. We apply a generalised Poisson-Jensen formula (due to Nevanlinna) to show that the investigated function has only a finite number of zeros in the interior of the unit disk. We further show, that the Li coefficients $\lambda_n = \sum_{\rho} (1 - (1 - 1/\rho)^n)$ are positive for $n > 10^{24}$, and discuss consequences.

1. INTRODUCTION

In 1997, Xian-Jin Li derived an unexpected criterion equivalent to the Riemann Hypothesis [13]. He showed that the positivity of the sequence of (real) numbers, the Li coefficients

$$(1.1) \quad \lambda_n = \frac{1}{\Gamma(n)} \frac{d^n}{ds^n} [s^{n-1} \log(\xi(s))]_{s=1},$$

where $\xi(s)$ is defined as

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

is equivalent to the Riemann Hypothesis.

Li derived his results in an interesting setting. He applied to $\xi(s)$ the transformation $z = 1 - \frac{1}{s}$, which maps the critical line to the unit circle, and the half-plane right to the critical line to the unit disk. In other words, he investigated

$$\phi(z) = \xi\left(\frac{1}{1-z}\right).$$

The λ_n are then the Taylor-coefficients of the logarithmic derivative of $\phi(z)$

$$(1.2) \quad \frac{\phi'(z)}{\phi(z)} = \sum_{n=0}^{\infty} \lambda_{n+1} z^n.$$

By applying the Hadamard product to $\phi(z)$, Li derived an alternative representation of the λ_n :

$$(1.3) \quad \lambda_n = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho}\right)^n \right]$$

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Here, the sum runs over the nontrivial zeros of $\zeta(s)$, paired and counted with multiplicity, and \sum_{ρ} has to be understood as $\lim_{T \rightarrow \infty} \sum_{\rho: |\rho| < T}$.

In a largely unknown work of 1992 [11], J.B. Keiper obtained similar results. Especially he defined coefficients similar to Li's (Keiper's coefficients equal $\frac{1}{n}\lambda_n$), but did not state as clearly as Li the equivalence of the positivity of the sequence of λ_n to Riemann Hypothesis.

E. Bombieri and J. Lagarias [1] further investigated Li's criterion and could show, that it is closely connected to the explicit formulae of prime number theory, and implicitly gave a splitting of Li coefficients into ([1], Theorem 2)

$$(1.4) \quad \lambda_n = \left[1 - (\log(4\pi) + \gamma) \frac{n}{2} - \sum_{j=2}^n (-1)^{j-1} \binom{n}{k} (1 - 2^{-j}) \zeta(j) \right] - \sum_{j=1}^n \binom{n}{k} \eta_{j-1}.$$

Here η_n denote coefficients which are closely connected to the Stieltjes constants γ_n ($\gamma_0 = \gamma$, the Euler constant)

$$(1.5) \quad \gamma_n := \frac{(-1)^n}{n!} \lim_{x \rightarrow \infty} \left(\sum_{k \leq x} \frac{1}{k} (\log(k))^n - \frac{(\log(x))^{n+1}}{n+1} \right).$$

$$(1.6) \quad \eta_n := \frac{(-1)^n}{n!} \lim_{x \rightarrow \infty} \left(\sum_{k \leq x} \frac{\Lambda(k)}{k} (\log(k))^n - \frac{(\log(x))^{n+1}}{n+1} \right),$$

where $\Lambda(k)$ is the von Mangoldt function, defined as

$$\Lambda(k) = \begin{cases} \log(p) & : \text{ k is a prime } p \text{ or any power of a prime, } p^n \\ 0 & : \text{ otherwise} \end{cases}$$

We have the corresponding series representations of $\zeta(s)$ resp. $\zeta'(s)/\zeta(s)$ (see e.g. [1]) :

$$(1.7) \quad \zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \gamma_k (s-1)^k \text{ and}$$

$$(1.8) \quad -\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + \sum_{k=0}^{\infty} \eta_k (s-1)^k$$

In his computation of the Li coefficients for $n \leq 3300$ in [14], K. Maslanka coined the handy terms 'trend' and 'oscillations' for the first and second part of (1.4), and denoted them by $\bar{\lambda}_n$ and $\tilde{\lambda}_n$ respectively

$$(1.9) \quad \lambda_n =: \bar{\lambda}_n + \tilde{\lambda}_n.$$

His calculations showed that the trend part $\bar{\lambda}_n$ is strictly growing with n , while the oscillatory part $\tilde{\lambda}_n$ is < 20 in absolute value, over the range of his computations.

So an apparent strategy for proving Riemann Hypothesis is, to show that $\bar{\lambda}_n \geq |\tilde{\lambda}_n|$ for all n . Maslanka was well aware of this strategy, and gave approximative expressions for $\bar{\lambda}_n$ ([14], (2.7)).

A. Voros [20] could further precise the behavior of $\bar{\lambda}_n$, giving the asymptotic expansion ([20], (24))

$$(1.10) \quad \bar{\lambda}_n \sim \frac{n}{2}(\log(n) - 1 - \gamma - \log(2\pi)) + \frac{3}{4} - \sum_{k=1}^{\infty} \frac{B_{2k}}{4k} n^{1-2k} \quad (n \rightarrow \infty)$$

(B_{2k} being the Bernoulli numbers). For our purpose it is enough to have the lower bounds on $\bar{\lambda}_n$ as shown by Coffey ([3], Theorem 2)

$$(1.11) \quad \bar{\lambda}_n \geq \frac{n}{2} \log(n) + (\gamma - 1) \frac{n}{2} + \frac{1}{2}.$$

Regarding the oscillations $\tilde{\lambda}_n$, less progress has been made. The deceptively simple expression

$$(1.12) \quad \tilde{\lambda}_n = - \sum_{j=1}^n \binom{n}{k} \eta_{j-1}$$

appears to be rather intractable. Coffey in [3] could give upper bounds for $|\eta_n|$ and proved the alternation in sign for η_n [4]. Maslanka, in [14] and [15], gave explicit expressions of $\tilde{\lambda}_n$ in terms of the Stieltjes constants (1.6), not allowing for an easy estimation of $|\tilde{\lambda}_n|$ however.

In this paper we will give an upper bound to $|\tilde{\lambda}_n|$, and prove that *the Li coefficients λ_n are nonnegative for all $n > 10^{24}$.*

Before, we need some other results however.

2. REQUIRED RESULTS FROM NEVANLINNA THEORY

Let

$$(2.1) \quad c_k(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{i\theta})| e^{-ik\theta} d\theta$$

be the Fourier coefficients of a meromorphic function $\log|f(re^{i\theta})|$, i.e.

$$(2.2) \quad \log|f(re^{i\theta})| = \sum_{k=-\infty}^{\infty} c_k(r, f) e^{ik\theta},$$

From [19] we then have the following

Theorem 2.1. *If $f(z)$ is meromorphic in $|z| \leq R$, with $f(0) \neq 0, \infty$, and $\{z_n\}$ the sequence of zeros, $\{w_n\}$ the sequence of poles of f , and if $\log(f(z)) = \sum_{k=0}^{\infty} \alpha_k z^k$ near $z = 0$, then for $0 < r \leq R$, we have*

$$(2.3) \quad \log|f(re^{i\theta})| = \sum_{k=-\infty}^{\infty} c_k(r, f) e^{ik\theta},$$

where the $c_k(r, f)$ are given by

$$(2.4) \quad c_0(r, f) = \log|f(0)| + \sum_{|z_n| \leq r} \log \frac{r}{|z_n|} - \sum_{|w_n| \leq r} \log \frac{r}{|w_n|}$$

For $k = 1, 2, 3, \dots$,

$$(2.5) \quad c_k(r, f) = \frac{1}{2} \alpha_k r^k + \frac{1}{2k} \sum_{|z_n| \leq r} \left[\left(\frac{r}{z_n} \right)^k - \left(\frac{\bar{z}_n}{r} \right)^k \right] \\ - \frac{1}{2k} \sum_{|z_n| \leq r} \left[\left(\frac{r}{w_n} \right)^k - \left(\frac{\bar{w}_n}{r} \right)^k \right]$$

For $k = 1, 2, 3, \dots$,

$$(2.6) \quad c_k(r, f) = \overline{c_k(r, f)}.$$

Remark 2.2. The theorem is also called *Fundamental Theorem of Nevanlinna Theory*, or *Poisson-Jensen-Nevanlinna formula*. For a proof see e.g. [19] (Lemma 4.2), [9] (Theorem 2.4), or Nevanlinna's original work [17].

In addition to Theorem 2.1, we need some standard concepts from *Nevanlinna theory*. Define

$$(2.7) \quad N(r, f) := \int_0^r \frac{n(t, f)}{t} dt,$$

where $n(r, f)$ denotes the number of poles of f in the disk $\{z : |z| \leq r\}$, and

$$(2.8) \quad m(r, f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where $\log^+(x) = \max(\log(x), 0)$. The *Nevanlinna characteristic*, a means to measure the growth of the function f , is then defined as

$$(2.9) \quad T(r, f) := m(r, f) + N(r, f).$$

A basic result of Nevanlinna theory is

$$(2.10) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} |\log|f(re^{i\theta})|| d\theta \leq 2T(r, f) + \log|f(0)|.$$

3. REQUIRED RESULTS FROM THE THEORY OF ZETA-FUNCTION

K. Ford in [8] could derive the following explicit estimate for the modulus of $\zeta(s)$ in the critical strip:

Theorem 3.1. *The inequality*

$$(3.1) \quad |\zeta(\sigma + it)| \leq At^{B(1-\sigma)^{3/2}} (\log(t))^{2/3} \quad (t \geq 3, \frac{1}{2} \leq \sigma \leq 1)$$

holds with $B = 4.45$ and $A = 76.2$.

Ford's work is an improvement of former results of Richert [18] and Cheng [5].

We also will need some estimations of the Stieltjes constants (1.5). Nan-You and Williams [16] give the following bounds:

$$|\gamma_n| \leq \begin{cases} \frac{2(2n)!}{n^{n+1}(2\pi)^n n!} & n \text{ odd} \\ \frac{4(2n)!}{n^{n+1}(2\pi)^n n!} & n \text{ even} \end{cases}$$

We adjust these estimation a bit for our needs. Taking ([6] (Lemma 12))

$$\frac{(2n)!}{n!} \leq \sqrt{2} \left(\frac{4n}{e} \right)^n \text{ for } k \geq 1$$

we get

$$(3.2) \quad |\gamma_n| \leq \frac{4(2n)!}{n^{n+1}(2\pi)^n n!} = \frac{4(2n)!}{n \left(\frac{5\pi n}{9}\right)^n \left(\frac{18}{5}\right)^n n!} \leq \frac{4\sqrt{2}}{n \left(\frac{18}{5}\right)^n} \left(\frac{36}{5e\pi}\right)^n \leq \frac{1}{3.6^{n+1}}$$

for $n \geq 7$.

4. ON THE POSITIVITY OF LI COEFFICIENTS

It is natural to investigate the function $(s-1)\zeta(s)$ as this has the same relation to $\tilde{\lambda}_n$ as $\xi(s)$ has to λ_n , e.g.

$$(4.1) \quad \tilde{\lambda}_n = \frac{1}{\Gamma(n)} \frac{d^n}{ds^n} [s^{n-1} \log((s-1)\zeta(s))]_{s=1},$$

cf. (1.1). Again we consider this function under the Möbius transformation $z = 1 - \frac{1}{s}$, i.e. we are interested in

$$(4.2) \quad f(z) = \frac{z}{1-z} \zeta\left(\frac{1}{1-z}\right).$$

$f(z)$ is analytic in the unit disk. Similar to (1.2) we also have

$$(4.3) \quad \frac{f'(z)}{f(z)} = \frac{d}{dz} \log(f(z)) = \sum_{k=1}^{\infty} \tilde{\lambda}_{n+1} z^n.$$

A necessary and sufficient condition for Riemann Hypothesis being true is that $f'(z)/f(z)$ is analytic in the unit disk. So we cannot assume of $\log(f(z))$ to be analytic in the unit disk. We can safely say however, that $f'(z)/f(z)$ is analytic near the origin (actually it is analytic at least in $\{z : |z| \leq 1 - 10^{-23}\}$ from what we know numerically about the zeros of $\zeta(s)$, see also the discussion in Remark 4.6). We therefore can integrate the series (4.3) term-by-term, to get

$$(4.4) \quad \log(f(z)) = \sum_{k=1}^{\infty} \frac{\tilde{\lambda}_{n+1}}{n+1} z^{n+1} = \sum_{k=0}^{\infty} \frac{\tilde{\lambda}_n}{n} z^n,$$

which is also analytic near the origin.

Next we want to get some information on the growth of $f(z)$ in the unit disk. As $f(z)$ is unbounded, we will investigate $(z-1)^5 f(z) = z(1-z)^4 \zeta\left(\frac{1}{1-z}\right)$. First we show

Theorem 4.1. *On the critical line,*

$$(4.5) \quad \left| \frac{\zeta(s)}{s^4} \right| < 23.4080.$$

Remark 4.2. Apparently, the desired result would be the value at $s = \frac{1}{2}$, i.e. $\left| \frac{\zeta(s)}{s^4} \right| < 16 \left| \zeta\left(\frac{1}{2}\right) \right| \approx 23.3657$. The given estimation is good enough for our purpose however.

Proof. We will do the estimation in three steps, first for $t \geq 3$, then for $0.5 \leq t \leq 3$, and finally for $t \leq 0.5$.

From (3.1) we have, for $s = \frac{1}{2} + it$ and $t \geq 3$,

$$(4.6) \quad \begin{aligned} \left| \frac{s-1}{s^5} \zeta(s) \right| &= \left| \frac{\zeta(s)}{s^4} \right| \leq \frac{76.2t^{4.45(1/2)^{3/2}} \log^{2/3}(t)}{t^4} \\ &\leq \frac{76.2t^{1.6} \log^{2/3}(t)}{t^4} = 76.2t^{-2.4} \log^{2/3}(t) \leq 5.8089. \end{aligned}$$

The function on the right hand side of the above equation is strictly decreasing with growing t , and therefore has its maximum at $t = 3$ with a value ≤ 5.8089 , which is well below the bound given in our theorem.

Next we will do the estimation for $0.5 \leq t \leq 3$. For $t \leq 3$, we investigate the representation (1.10) of $\zeta(s)$ as a series about 1:

$$(s-1)\zeta(s) = 1 + \sum_{k=0}^{\infty} \gamma_k (s-1)^{k+1}.$$

Using the estimation (3.7), $|\gamma_n| \leq \frac{1}{3.6^{n+1}}$, which is valid for $n \geq 7$, we get

$$(4.7) \quad |(s-1)\zeta(s)| \leq 1 + \sum_{k=0}^{\infty} \left| \frac{s-1}{3.6} \right|^{k+1} = \sum_{k=0}^{\infty} \left| \frac{s-1}{3.6} \right|^k = \frac{18}{18-5|s-1|}$$

for $|s-1| < 3.6$, or if $s = \frac{1}{2} + it$, for $t < \sqrt{3.6^2 - (\frac{1}{2})^2} \approx 3.5651$, i.e. for $t > 3$.

For a more detailed estimation of the series, we use the maximal modulus for points on the critical line which we are interested in currently, i.e. for $t \leq 3$, which is $|\frac{1}{2} + 3it| = \sqrt{3^2 + (\frac{1}{2})^2} \approx 3.0414$. We get

$$\begin{aligned} |(s-1)\zeta(s)| &\leq |1 + \gamma_0(s-1) + \gamma_1(s-1)^2| + \sum_{k=2}^6 |\gamma_k| |s-1|^{k+1} + \sum_{k=7}^{\infty} \left| \frac{s-1}{3.6} \right|^{k+1} \\ &= |1 + \gamma_0(s-1) + \gamma_1(s-1)^2| + \sum_{k=2}^6 |\gamma_k| |s-1|^{k+1} + \left| \frac{s-1}{3.6} \right|^6 \sum_{k=0}^{\infty} \left| \frac{s-1}{3.6} \right|^k \\ &= |1 + \gamma_0(s-1) + \gamma_1(s-1)^2| + \sum_{k=2}^6 |\gamma_k| |s-1|^{k+1} + \left| \frac{s-1}{3.6} \right|^6 \frac{18}{18-5|s-1|} \end{aligned}$$

We put $s = \frac{1}{2} + it$:

$$\begin{aligned} \left| \left(-\frac{1}{2} + it \right) \zeta \left(\frac{1}{2} + it \right) \right| &\leq \left| 1 + \gamma_0 \left(-\frac{1}{2} + it \right) + \gamma_1 \left(-\frac{1}{2} + it \right)^2 \right| \\ &\quad + \sum_{k=2}^6 |\gamma_k| |3.1|^{k+1} + \left| \frac{3.1}{3.6} \right|^6 \frac{18}{18 - 5 \cdot 3.1} \\ &\leq \left| 1 + \gamma_0 \left(-\frac{1}{2} + it \right) + \gamma_1 \left(-\frac{1}{2} + it \right)^2 \right| + 0.2105 + 2.9355 \\ &= \left| 1 + \gamma_0 \left(-\frac{1}{2} + it \right) + \gamma_1 \left(-\frac{1}{2} + it \right)^2 \right| + 3.1460 \end{aligned}$$

Now, for $0.5 \leq t \leq 3$,

$$\frac{\left| 1 + \gamma_0 \left(-\frac{1}{2} + it \right) + \gamma_1 \left(-\frac{1}{2} + it \right)^2 \right| + 3.1460}{\left| \frac{1}{2} + it \right|^5}$$

is a strictly decreasing polynomial with maximal value at $t = 0.5$, which approximately evaluates to 22.0661, just below our desired bound.

Finally, we handle the case $0 \leq t \leq \frac{1}{2}$. The remainder term of our estimation now evaluates to, using the maximal modulus, at $t = \frac{1}{2}$, $|\frac{1}{2} + it - 1| \leq |\frac{1}{2} + i\frac{1}{2}| = \sqrt{\frac{1}{2}} \approx 0.7071$,

$$\sum_{k=2}^6 |\gamma_k| |0.71|^{k+1} + \left| \frac{0.71}{3.6} \right|^6 \frac{18}{18 - 5 \cdot 0.71} \approx 0.001838 + 0.000073 = 0.001903.$$

This yields a maximum value for

$$\frac{\left| 1 + \gamma_0 \left(-\frac{1}{2} + it \right) + \gamma_1 \left(-\frac{1}{2} + it \right)^2 \right| + 0.001903}{\left| \frac{1}{2} + it \right|^5}$$

at $t = 0$, of 23.4080, as stated.

Symmetry of $\zeta(s)$ about the real axis gives the same results for negative t . □

Corollary 4.3. For $|z| \leq 1$,

$$(4.8) \quad \left| z(1-z)^4 \zeta \left(\frac{1}{1-z} \right) \right| \leq 23.4080.$$

Proof. From the maximum modulus principle it is enough to find the maximum on the unit circle, which is the image of the critical line under the transformation $z = 1 - 1/s$. Inverting this transformation, $(z-1)^5 f(z)$ maps back to $\frac{s-1}{s^5} \zeta(s)$, for which, on the critical line,

$$\left| \frac{s-1}{s^5} \zeta(s) \right| = \left| \frac{\zeta(s)}{s^4} \right|$$

The statement follows then from Theorem 4.1. □

Corollary 4.4. *There is only a finite number of nontrivial zeros of $\zeta(s)$ violating Riemann Hypothesis. More precisely, there are not more than 374 nontrivial zeros with imaginary part $\neq \frac{1}{2}$.*

Proof. We will apply Theorem 2.1 to $f(z) = \frac{z}{1-z}\zeta(\frac{1}{1-z})$. Regarding the preliminaries, $f(z)$ is analytic in the unit disk, and therefore meromorphic in $|z| \leq R = 1$, with $f(0) = 1 \neq 0, \infty$. $f(z)$ has no poles in the unit disk, and any zero would be an exception to Riemann Hypothesis.

Therefore we can apply (2.4), which states

$$c_0(r, f) = \log|f(0)| + \sum_{|z_n| \leq r} \log \frac{r}{|z_n|} - \sum_{|w_n| \leq r} \log \frac{r}{|w_n|}$$

or

$$(4.9) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{i\theta})| d\theta = \sum_{|z_n| \leq r} \log \frac{r}{|z_n|},$$

where we have dropped the term regarding the poles, which is not applicable. This is of course Jensen's formula in its standard form.

Further, we have from Corollary 4.3

$$|(1-z)^5 f(z)| \leq 23.4080$$

This means, for the integral in (4.10), we have the estimation

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{i\phi})| d\phi &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left(\frac{23.4080}{|1-re^{i\phi}|^5} \right) d\phi \\ &\leq \log(23.4080) + \frac{1}{\pi} \int_0^{\pi} \log|\csc(\phi)|^5 d\phi \\ &= \log(23.4080) + \log(32) \\ &= \log(749.056) \\ &\approx 6.6188 \end{aligned}$$

(see [7] for the estimation to $\csc(\phi)$). Putting this into (4.10), we get

$$\sum_{|z_n| \leq r} \log \frac{r}{|z_n|} \leq \log(749.056)$$

or

$$(4.10) \quad \prod_{|z_n| \leq r} \frac{r}{|z_n|} \leq 749.056$$

Note, that we can chose r arbitrarily close to 1. If we gradually increase r , we finally will cover all zeros of $\zeta(s)$ off the critical line (otherwise there would be an infinite number of them, which would mean that $f(z) = 0$ identically, from the

Identity theorem. From numerical data (e.g. [10]) we know, that there are no exceptional zeros $\rho = \sigma + it$ up to $t = 2445999556030 \approx 2.45 \cdot 10^{12}$. This means for optional zeros z_n to have a modulus close to 1 by a magnitude $< 10^{-20}$. I.e. each of the factors on the left is a negligible amount bigger than 1. As, for symmetry reasons, any exceptional root is counted twice, this finally means that there are no more than $\lfloor \frac{749.056}{2} \rfloor = 374$ exceptional zeros. \square

By applying further parts of Theorem 2.1 to $f(z)$, we can prove the following

Theorem 4.5. *For $n > 10^{24}$ all Li coefficients λ_n are positive.*

Proof. Preconditions of Theorem 2.1 are fulfilled for $f(z) = \frac{z}{1-z} \zeta(\frac{1}{1-z})$ (see the proof of Corollary 4.3). Further, from (4.4),

$$\log(f(z)) = \sum_{k=0}^{\infty} \frac{\tilde{\lambda}_k}{k} z^k$$

near $z = 0$. (2.5) now states, for $k = 1, 2, 3, \dots$,

$$\begin{aligned} c_k(r, f) &= \frac{1}{2} \frac{\tilde{\lambda}_k}{k} r^k + \frac{1}{2k} \sum_{|z_n| \leq r} \left[\left(\frac{r}{z_n} \right)^k - \left(\frac{\bar{z}_n}{r} \right)^k \right] \\ &\quad - \frac{1}{2k} \sum_{|w_n| \leq r} \left[\left(\frac{r}{w_n} \right)^k - \left(\frac{\bar{w}_n}{r} \right)^k \right] \end{aligned}$$

or, dropping again the expressions regarding the poles

$$\frac{1}{2} \frac{\tilde{\lambda}_k}{k} r^k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{i\theta})| e^{-ik\theta} d\theta - \frac{1}{2k} \sum_{|z_n| \leq r} \left[\left(\frac{r}{z_n} \right)^k - \left(\frac{\bar{z}_n}{r} \right)^k \right]$$

Then

$$\begin{aligned} \left| \frac{1}{2} \frac{\tilde{\lambda}_k}{k} r^k \right| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{i\theta})| e^{-ik\theta} d\theta - \frac{1}{2k} \sum_{|z_n| \leq r} \left[\left(\frac{r}{z_n} \right)^k - \left(\frac{\bar{z}_n}{r} \right)^k \right] \right| \\ &\leq \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{i\theta})| e^{-ik\theta} d\theta \right| + \left| \frac{1}{2k} \sum_{|z_n| \leq r} \left[\left(\frac{r}{z_n} \right)^k - \left(\frac{\bar{z}_n}{r} \right)^k \right] \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\log|f(re^{i\theta})|| d\theta + O(10^{-50}) \end{aligned}$$

or

$$(4.11) \quad \left| \tilde{\lambda}_k \right| \leq \frac{2k}{r^k} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\log|f(re^{i\theta})|| d\theta + O(10^{-50}) \right)$$

As r is arbitrarily close to 1 we can ignore the factor $1/r^k$.

Now we need to compute the Nevanlinna characteristic for $f(z)$, acc. to (2.7) - (2.9). We already have, from the estimations in to proof to Corollary 4.4

$$\begin{aligned}
m(r, f) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln^+ |f(re^{i\phi})| d\phi \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\phi})| d\phi \\
&\leq \log(23.4080) + \log(32) = \log(749.056) \approx 6.6188.
\end{aligned}$$

Further, as $f(z)$ has no poles in the unit disk, $n(t, f) = 0$, and therefore

$$N(r, f) = \int_0^r \frac{n(t, f)}{t} dt = 0$$

From that, the Nevanlinna characteristic computes to

$$T(r, f) = m(r, f) + N(r, f) \leq \log(749.056) \approx 6.6188.$$

From (2.10) we finally get the estimation

$$|c_k(r, f)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\log |f(re^{i\theta})|| d\theta \leq 2T(r, f) \approx 13.2376$$

This means, by (4.11), that

$$(4.12) \quad \left| \tilde{\lambda}_k \right| \leq 2k \cdot 13.2376 = 26.4752 \cdot k$$

With this result, we can continue to estimate the Li coefficients λ_k . We have (1.12):

$$\lambda_k = \bar{\lambda}_k + \tilde{\lambda}_k$$

With Coffey's lower bound estimation (1.14)

$$\bar{\lambda}_k \geq 0.5(1 + k(\log(k) - (1 - \gamma)k)) \approx 0.5 + 0.5k \cdot \log(k) - 0.2114k > 0$$

we get

$$\begin{aligned}
\lambda_k &\geq \bar{\lambda}_k - |\tilde{\lambda}_k| \\
&\geq 0.5 + 0.5k \cdot \log(k) - 0.2114k - 26.4752k \\
&= 0.5k \cdot \log(k) - 26.6866k
\end{aligned}$$

which is nonnegative for $k \geq e^{53.3732}$, or $k > 10^{23.1797}$. In other words, the Li criterion is fulfilled for all $k > 10^{24}$. \square

Remark 4.6. The zeros of $\zeta(s)$ are known up to a height of $Z_{max} = 2.45 \cdot 10^{12}$ on the critical line. It is known ([20], p. 8, [2], p. 441), that a violation to the positivity of λ_n could only occur at $n > Z_{max}^2 = 5.67 \cdot 10^{24}$. As we could not find a detailed account of the argument, we try elaborate this here.

Assume there would be a zero ρ of $\zeta(s)$ off the critical line, just above the known zeros, e.g. $\rho = \frac{1}{2} + \delta + i \cdot 2.4 \cdot 10^{12}$, where $|\delta| < \frac{1}{2}$ (there are no zeros outside the critical strip). Then this zero would be mapped by $z = 1 - 1/s$ to a point close to 1, with coordinates:

$$\begin{aligned}
 (4.13) \quad 1 - \frac{1}{\rho} &= \left(1 - \frac{1/2 + \delta}{(1/2 + \delta)^2 + 2.4^2 \cdot 10^{24}} \right) + i \frac{2.4 \cdot 10^{12}}{(1/2 + \delta)^2 + 2.4^2 \cdot 10^{24}} \\
 &\approx \left(1 - \frac{1/2 + \delta}{2.4^2 \cdot 10^{24}} \right) + i \frac{1}{2.4 \cdot 10^{12}},
 \end{aligned}$$

Now consider the Li coefficient

$$\lambda_n = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho} \right)^n \right].$$

The contributions to λ_n from the zeros on the critical line are positive (see the proof of Li for his criterion, in [13]). Regarding the contribution of ρ , we actually have four roots to consider: $\frac{1}{2} \pm \delta \pm i \cdot 2.4 \cdot 10^{12}$, due to the symmetry properties of $\zeta(s)$. The pair $\frac{1}{2} + \delta \pm i \cdot 2.4 \cdot 10^{12}$ maps to the interior of the disk, with modulus < 1 , and contributes a positive value to λ_n . The only chance we have to get λ_n negative comes from $\frac{1}{2} - \delta \pm i \cdot 2.4 \cdot 10^{12}$, which map to just outside of the unit disk, therefore having a modulus > 1 . More precisely, this modulus is

$$\begin{aligned}
 (4.14) \quad \left| 1 - \frac{1}{\frac{1}{2} - \delta \pm i \cdot 2.4 \cdot 10^{12}} \right| &= \sqrt{\left(1 - \frac{1/2 - \delta}{2.4^2 \cdot 10^{24}} \right)^2 + \left(\frac{1}{2.4 \cdot 10^{12}} \right)^2} \\
 &= \sqrt{\frac{(2.4^2 \cdot 10^{24} - (1/2 - \delta))^2}{2.4^4 \cdot 10^{48}} + \frac{2.4^2 \cdot 10^{24}}{2.4^4 \cdot 10^{48}}} \\
 &= \sqrt{\frac{2.4^4 \cdot 10^{48} + 2\delta \cdot 2.4^2 \cdot 10^{24} + (1/2 - \delta)^2}{2.4^4 \cdot 10^{48}}} \\
 &\approx \sqrt{1 + \frac{2\delta}{2.4^2 \cdot 10^{24}}} \\
 &\geq 1 + \frac{\delta}{2.4^2 \cdot 10^{24}}
 \end{aligned}$$

Applying again Bernoulli's estimation, and ignoring $Arg(\rho)$ for a moment, i.e. assuming $1 - (1 - 1/\rho)$ being real, we get

$$1 - \left(1 - \frac{1}{\rho} \right)^n \leq 1 - \left(1 + \frac{\delta}{2.4^2 \cdot 10^{24}} \right)^n \leq 1 - \exp\left(\frac{n/2}{2.4^2 \cdot 10^{24}} \right)$$

or generally,

$$(4.15) \quad 1 - \left(1 - \frac{1}{\rho} \right)^n \leq 1 - \exp\left(\frac{n/2}{\text{Im}(\rho)^2} \right)$$

We see already, that n has to be at least of order $\text{Im}(\rho)^2$ or $5.7 \cdot 10^{24}$ to achieve a negative contribution to λ_n .

But counting to the sum of zeros is not the modulus, but (the real part of) the pair $(1 - 1/\rho)^n + (1 - 1/\bar{\rho})^n$, which is approximately the modulus times $2\cos(n \cdot Arg(1 - 1/\rho)) \approx 2\cos(n \cdot \arcsin(\text{Im}(1 - 1/\rho))) \approx 2\cos(n \cdot \text{Im}(1 - 1/\rho)) \approx 2 - 2n \cdot \text{Im}(1 - 1/\rho)$. So we get

$$\begin{aligned} 2 - \left(1 - \frac{1}{\rho}\right)^n - \left(1 - \frac{1}{\bar{\rho}}\right)^n &\leq 2 - 2 \left(1 - \frac{n}{2.4 \cdot 10^{12}}\right) \left(1 - \exp\left(\frac{n/2}{2.4^2 \cdot 10^{24}}\right)\right) \\ &= \frac{n^2 - 2n \cdot 2.4^2 \cdot 10^{24}}{2.4^3 \cdot 10^{36}} - \exp\left(\frac{n/2}{2.4^2 \cdot 10^{24}}\right) \end{aligned}$$

Finally, we have to consider the size of the trend $\bar{\lambda}_n$, which is $\approx \frac{1}{2}k \cdot \log(k)$ and which has to be compensated in order to get a negative λ_n . This means we have to resolve

$$\frac{1}{2}n \cdot \log(n) + \frac{n^2 - 2n \cdot 2.4^2 \cdot 10^{24}}{2.4^3 \cdot 10^{36}} - \exp\left(\frac{n/2}{2.4^2 \cdot 10^{24}}\right) < 0$$

for n , which yields the condition $n > 7.5 \cdot 10^{26}$ (the evaluation also shows that we can ignore the contribution of $\text{Arg}(\rho)$, as is to be expected by the properties of the exponential function).

For our purpose it is safe to say that, given an exceptional zero with imaginary part $> 2.4 \cdot 10^{12}$, n has to be $> 10^{25}$ for λ_n to become negative.

We now can state

Theorem 4.7. *Riemann hypothesis is true, i.e. all nontrivial zeros of $\zeta(s)$ lie on the critical line $\text{Re}(s) = \frac{1}{2}$.*

Proof. By Theorem 4.5 and Remark 4.6, Li's criterion is proved, which is equivalent to Riemann Hypothesis [13]. \square

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