# The Canonical Commutation Relation derives from the Homogeneity Symmetry, but needs Accidental Coincident Scalings to be Unitary

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27th December 2017

**Abstract** Textbook theory says that the Canonical Commutation Relation derives from the homogeneity of space. This paper shows that additionally, an *accidental coincidence of scales* is needed, as extra information, without which the Canonical Commutation Relation is left non-unitary and broken. This single counter-example removes symmetry, as intrinsic ontological reason, for axiomatically imposing unitarity (or self-adjointness) — *by Postulate* — on quantum mechanical systems.

**Keywords** foundations of quantum theory, quantum mechanics, wave mechanics, Canonical Commutation Relation, symmetry, homogeneity of space, unitary.

## 1 Homogeneity of Space and Wave Mechanics

The *Canonical Commutation Relation*

$$\mathbf{px} - \mathbf{xp} = -i\hbar$$

embodies core algebra at the heart of wave mechanics. The professed significance of this relation, with general acceptance by quantum theorists, is that it represents the homogeneity of space as being unitary. In this chapter, I re-examine and scrutinise the Canonical Relation's derivation and establish that the homogeneity symmetry is itself *not* unitary. And in consequence establish that the Canonical Commutation Relation does not, itself, faithfully represent homogeneity, but contains other (unitary) information also.

Imposing homogeneity on a system is identical to imposing a null physical or geometrical effect, under arbitrary translation of reference frame. To formulate this arbitrary translation, resulting in null effect, the principle we invoke is *Form Invariance*. This is the concept, from relativity, that symmetry transformations leave (physical) formulae fixed in *form*, though *values* may alter [1]. In the case at hand, the relevant formula whose form is held fixed is the eigenvalue equation for position:

$$\mathbf{x}\left|f_{\mathsf{x}}\left(x\right)\right\rangle = \mathsf{x}\left|f_{\mathsf{x}}\left(x\right)\right\rangle . \tag{1}$$

The san-serif $\mathsf{x}$, here, is a label for $f_{\mathsf{x}}$ whose eigenvalue is $\mathsf{x}$. The variable $x$ (curly) is the function domain. The use of two different variables, here, may seem unusual and pointless. In fact, logically they are different. $\mathsf{x}$ is quantified *existentially* but $x$ is quantified *universally.*

With form held fixed, as the reference system is displaced, variation in the position operator $\mathbf{x}$ determines a group relation, representing the homogeneity symmetry. Under arbitrarily small displacements, this group corresponds to a linear algebra representing homogeneity locally (Lie group and Lie algebra). To maintain the form of (1), under translation, the basis $\left|f_{\mathsf{x}}\right\rangle$ is cleverly managed: while the translation transforms the basis from $\left|f_{\mathsf{x}}\right\rangle$ to $\left|f_{\mathsf{x}-\epsilon}\right\rangle$, a similarity transformation is also applied, chosen to revert $\left|f_{\mathsf{x}-\epsilon}\right\rangle$ back to $\left|f_{\mathsf{x}}\right\rangle$. In this way $\left|f_{\mathsf{x}}\right\rangle$ is held static. We see below that, actually, similarity transforms can be found only for a certain class of functions: $\left\{\psi_{\mathsf{x}} \in L^1\right\} \subset \left\{f_{\mathsf{x}}\right\}$.

The similarity transformations are the one-parameter subgroup of the general

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$$\begin{tikzcd}
\mathbf{x}\,|f_{\mathsf{x}}\rangle = \mathsf{x}\,|f_{\mathsf{x}}\rangle \arrow[r, "\text{translation}", "O_{\mathsf{x}}\to O_{\mathsf{x}'}"'] \arrow[d] & \mathbf{x}\,|f_{\mathsf{x}-\epsilon}\rangle = (\mathsf{x}+\epsilon)\,|f_{\mathsf{x}-\epsilon}\rangle \arrow[d, "\text{similarity}", "|f_{x-\epsilon}\rangle\to|\psi_{x-\epsilon}\rangle\to|\psi_{\mathsf{x}}\rangle"'] \\
\left(\mathsf{SxS}^{-1} - \epsilon\mathbb{1}\right)|\psi_{\mathsf{x}}\rangle = \mathbf{x}\,|\psi_{\mathsf{x}}\rangle & \left(\mathsf{SxS}^{-1} - \epsilon\mathbb{1}\right)|\psi_{\mathsf{x}}\rangle = \mathsf{x}\,|\psi_{\mathsf{x}}\rangle \arrow[l]
\end{tikzcd}$$

**Figure 1** Scheme of transformations. The bottom left hand formula is the resulting group relation.

linear group, $\mathsf{S}(\epsilon) \subset \mathsf{S} \in \mathsf{GL}(\mathbb{F})$, with the transformation parameter $\epsilon$ coinciding with the displacement parameter, and where $\mathbb{F}$ is any infinite field. The overall scheme of transformations is depicted in Figure 1.

In standard theory, textbook understanding is that $\mathsf{S}(\epsilon)$ is intrinsically and necessarily unitary, and it is in *that* unitarity where the Canonical Commutation Relation finds its unitary origins. And so, because its presence is thought *intrinsically necessary*, unitarity is imposed axiomatically on the theory, *by Postulate*. The upshot is that standard theory *imposes* Hilbert space on vectors $|f_{\mathsf{x}}\rangle$. This imposed unitarity is added information, extra to the information of homogeneity. In consequence, the underlying symmetry beneath wave mechanics is not homogeneity of space, but instead, a unitary subgroup of it.

As an experiment, I proceed by treating unitarity as a purely separate issue from homogeneity, allowing $\mathsf{S}(\epsilon)$ it's widest generality, so that the *whole information* of homogeneity (upto the general linear similarity transformation) is faithfully and genuinely conveyed through the theory.

The experiment begins with the position eigenvalue equation (1) being rewritten, in the form of a quantified proposition (2). From here on, all informal assumptions are to be shed, with the Dirac notation dropped to avoid any inference that vectors are intended as orthogonal, in Hilbert space, or equipped with a scalar product; none of these is implied.

Consider the eigenformula for position operator $\mathbf{x}$, eigenfunctions $f_{\mathsf{x}}$ and eigenvalues $\mathsf{x}$, seen from the reference frame $\mathsf{O}_{\mathsf{x}}$:

$$\forall x \exists \mathbf{x} \exists \mathsf{x} \exists f_{\mathsf{x}} \mid \quad \mathbf{x} f_{\mathsf{x}}(x) = \mathsf{x} f_{\mathsf{x}}(x) \tag{2}$$

***Translation:*** Applying the translation first. Under translation, homogeneity demands existence of an equally relevant reference frame $\mathsf{O}_{\mathsf{x}'}$ displaced arbitrarily through $\epsilon$. See Figure 2. *Form Invariance* guarantees a formula for $\mathsf{O}_{\mathsf{x}'}$ of the same form as that for $\mathsf{O}_{\mathsf{x}}$ in (2), thus:

$$\forall x' \exists \mathbf{x} \exists \mathsf{x}' \exists f'_{\mathsf{x}} \mid \quad \mathbf{x} f'_{\mathsf{x}}(x') = \mathsf{x}' f'_{\mathsf{x}}(x') \tag{3}$$

A relation for $\mathbf{x}$ is to be evaluated, so $\mathbf{x}$ is held static for all reference frames. The translation transforms position, thus:

$$\forall \epsilon \forall \mathsf{x}' \exists \mathsf{x} \mid \quad \mathsf{x} \mapsto \mathsf{x}' = \mathsf{x} + \epsilon \tag{4}$$

and transforms the function, thus:

$$\forall \epsilon \forall x' \forall f'_{\mathsf{x}} \exists f_{\mathsf{x}} \exists x \mid \quad f_{\mathsf{x}}(x) \mapsto f'_{\mathsf{x}}(x') = f_{\mathsf{x}-\epsilon}(x - \epsilon) \tag{5}$$

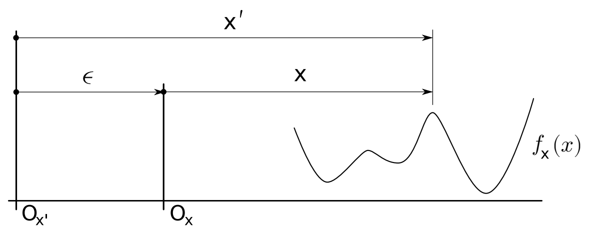


**Figure 2 Passive translation of a function** Two reference systems, $\mathsf{O}_{\mathsf{x}}$ and $\mathsf{O}_{\mathsf{x}'}$, arbitrarily displaced by $\epsilon$, individually act as reference systems for position of a function $f_{\mathsf{x}}$. If the $\mathsf{x}$-space is homogeneous, then regardless of the value of $\epsilon$, physics concerning this function is described by formulae whose form remains invariant, though values may change. **Note:** The function and reference frames are not epistemic; $f_{\mathsf{x}}$ is non-observable and $\mathsf{O}_{\mathsf{x}}$ and $\mathsf{O}_{\mathsf{x}'}$ are not observers.

Substituting (4) and (5) into (3) gives the translated formula:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists f_{\mathsf{x}} \mid \quad \mathbf{x} f_{\mathsf{x}-\epsilon}(x-\epsilon) = (\mathsf{x}+\epsilon) f_{\mathsf{x}-\epsilon}(x-\epsilon)\,. \tag{6}$$

***Similarity***: Applying the similarity transformation. This involves the one-parameter linear operator $\mathsf{S}_{(\epsilon)}$. Such an $\mathsf{S}_{(\epsilon)}$ exists only if there exists a space of functions $\psi_{\mathsf{x}}$, which is complete, normalisable, and not restricted to separable[1] functions, and which is also a subset of the translatable functions $f_{\mathsf{x}}$. See Figure 3. Logically, the act of assuming such an $\mathsf{S}_{(\epsilon)}$ hypothesises that such a class of functions does indeed exist. No such function space is guaranteed. Accordingly, the assertion of proposition (7) is newly assumed information entering the system.

$$\forall x \forall \epsilon \forall \psi_{\mathsf{x}-\epsilon} \exists \mathsf{S} \exists \psi_{\mathsf{x}} \mid \quad \mathsf{S}^{-1}_{(\epsilon)} \psi_{\mathsf{x}}(x) = \psi_{\mathsf{x}-\epsilon}(x-\epsilon)\,. \tag{7}$$

In standard theory, $\mathsf{S}_{(\epsilon)}$ is set unitary by the mathematician — axiomatically. The act of doing so is the application of the Quantum Postulate declaring the quantum system unitary and Hermitian. Restricting $\mathsf{S}_{(\epsilon)}$ in that way, in turn restricts the space of functions $\psi_{\mathsf{x}}$ to the Hilbert space $L^2$. But as it stands, $\mathsf{S}_{(\epsilon)}$ is a member of the one-parameter subgroup of the infinite dimensional, (non-unitary) general linear group $\mathsf{GL}(\mathbb{F})$. And with $\mathsf{S}_{(\epsilon)}$ a member of $\mathsf{GL}(\mathbb{F})$, it restricts $\psi_{\mathsf{x}}$ not to the Hilbert space $L^2$, but to the Banach space $L^1$.

The similarity transformation is formed, thus:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}^{-1}_{(\epsilon)} \psi_{\mathsf{x}}(x) = (\mathsf{x}+\epsilon)\psi_{\mathsf{x}}(x)\,.$$

Introducing the trivial eigenformula: $\forall \psi_{\mathsf{x}} \forall x \forall \epsilon \mid \epsilon \mathbb{1} \psi_{\mathsf{x}}(x) = \epsilon \psi_{\mathsf{x}}(x)$ and subtracting:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \left(\mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}^{-1}_{(\epsilon)} - \epsilon \mathbb{1}\right) \psi_{\mathsf{x}}(x) = \mathsf{x} \psi_{\mathsf{x}}(x)\,. \tag{8}$$

Now comparing the original position eigenformula (2) against the transformed one (8), we deduce the group relation for similarity transformed homogeneity:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \mathbf{x} \psi_{\mathsf{x}}(x) = \left(\mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}^{-1}_{(\epsilon)} - \epsilon \mathbb{1}\right) \psi_{\mathsf{x}}(x)\,. \tag{9}$$

From this group relation, the commutator for the *Lie algebra* is now computed. Because $\mathsf{S}_{(\epsilon)}$ is a one-parameter subgroup of $\mathsf{GL}(\mathbb{F})$, there exists a unique linear operator $\mathbf{g}$ for real parameters $\epsilon$, such that:

$$\forall \mathsf{S} \exists \mathbf{g} \mid \quad \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\epsilon \mathbf{g}} \tag{10}$$

Noting that homogeneity is totally independent of scale, an arbitrary scale factor $\eta$ is extracted, thus: $\forall \mathbf{g} \forall \eta \exists \mathbf{k} : \ \mathbf{g} = \eta \mathbf{k}$, implying:

$$\forall \eta \forall \mathsf{S} \exists \mathbf{k} \mid \quad \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\eta \epsilon \mathbf{k}} \tag{11}$$

$$\forall \eta \forall \mathsf{S} \exists \mathbf{k} \mid \quad \mathsf{S}^{-1}_{(\epsilon)} = \mathsf{S}_{(-\epsilon)} = \mathrm{e}^{-\eta \epsilon \mathbf{k}} \tag{12}$$

Substitution of (11) and (12) into (9) gives:

$$\begin{aligned}
&\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \exp(+\eta\epsilon\mathbf{k})\,\mathbf{x}\exp(-\eta\epsilon\mathbf{k})\,\psi_{\mathsf{x}}(x) = [\mathbf{x}+\epsilon\mathbb{1}]\,\psi_{\mathsf{x}}(x)\\
\Rightarrow\ &\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbb{1}+\eta\epsilon\mathbf{k}+\mathcal{O}\left(\epsilon^2\right)\right]\mathbf{x}\left[\mathbb{1}-\eta\epsilon\mathbf{k}+\mathcal{O}\left(\epsilon^2\right)\right]\psi_{\mathsf{x}}(x) = [\mathbf{x}+\epsilon\mathbb{1}]\,\psi_{\mathsf{x}}(x)\\
\Rightarrow\ &\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbf{x}+\eta\epsilon\mathbf{k}\mathbf{x}+\mathcal{O}\left(\epsilon^2\right)\right]\left[\mathbb{1}-\eta\epsilon\mathbf{k}+\mathcal{O}\left(\epsilon^2\right)\right]\psi_{\mathsf{x}}(x) = [\mathbf{x}+\epsilon\mathbb{1}]\,\psi_{\mathsf{x}}(x)\\
\Rightarrow\ &\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbf{x}+\eta\epsilon\mathbf{k}\mathbf{x}-\eta\epsilon\mathbf{x}\mathbf{k}+\mathcal{O}\left(\epsilon^2\right)\right]\psi_{\mathsf{x}}(x) = [\mathbf{x}+\epsilon\mathbb{1}]\,\psi_{\mathsf{x}}(x)\\
\Rightarrow\ &\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && [\mathbf{k}\mathbf{x}-\mathbf{x}\mathbf{k}]\,\psi_{\mathsf{x}}(x) = \left[\eta^{-1}\mathbb{1}-\mathcal{O}(\epsilon)\right]\psi_{\mathsf{x}}(x)
\end{aligned}$$

At the limit, as $\epsilon \to 0$, we have:

$$\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid \quad [\mathbf{k},\mathbf{x}]\,\psi_{\mathsf{x}}(x) = \eta^{-1}\mathbb{1}\psi_{\mathsf{x}}(x) \tag{13}$$

And by a proof similar to all that above, but conditional upon the existence of eigenfunctions $\chi_{\mathsf{k}}(k)$ of $\mathbf{k}$, and also upon the valid extraction of inverse scalar $\eta^{-1}$ at (11), rather than the scalar $\eta$:

$$\forall k \forall \eta \exists \mathsf{k} \exists \chi_{\mathsf{k}} \exists \mathbf{x} \exists \mathbf{k} \mid \quad [\mathbf{x},\mathbf{k}]\,\chi_{\mathsf{k}}(k) = \eta^{+1}\mathbb{1}\chi_{\mathsf{k}}(k)\,. \tag{14}$$

These formulae, (13) and (14) are two unengaged individuals. There is no assurance that they offer complementarity. As they stand they cannot substitute one into the other. Note that (13) is quantified $\forall\eta\exists\psi_{\mathsf{x}}$, and (14) is quantified $\forall\eta\exists\chi_{\mathsf{k}}$. And so

**Substitution involving quantified variables**

$$\begin{aligned}
&\forall\beta\underline{\forall\gamma}\exists\alpha \mid \ \alpha = \beta+\gamma\\
&\forall\beta\underline{\exists\gamma} \mid \ \gamma = \beta+\beta\\
\Rightarrow\ &\forall\beta\exists\alpha \mid \ \alpha = \beta+\beta+\beta
\end{aligned}$$

For logically dependent substitution, an existential quantifier of one proposition should be matched with a universal quantifier of the other. This is because, for this type of substitution coincidence is certain and not accidental. Matching quantifiers are underlined.

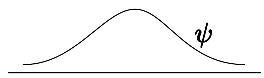


**Figure 3** The linear transformations $\mathsf{S}$ exist only for bounded $\psi_{\mathsf{x}}$, minimally the Banach space $L^1$.

[1] Separable means countable, as are the integers, as opposed to continuous, like the reals.

there is no guarantee that $\psi_{\mathsf{x}}$ and $\chi_{\mathsf{k}}$ ever coincide. For that, new information is needed, thus:

**Hypothesised coincidence:**

$$\begin{array}{rl} \textbf{either} & \forall\psi_{\mathsf{x}}\exists\chi_{\mathsf{k}} \mid \quad \psi_{\mathsf{x}} = \chi_{\mathsf{k}} \\ \textbf{alternatively} & \forall\chi_{\mathsf{k}}\exists\psi_{\mathsf{x}} \mid \quad \psi_{\mathsf{x}} = \chi_{\mathsf{k}} \end{array}$$

That new information forces conditionality on $\eta$. No longer is $\forall\eta$ possible, with $\eta$ restricted thus:

$$\eta^{-1} = -\eta$$

Now this might have seemed like a cheat because *I myself* picked the numbers that would do the job. But the point to notice is that no matter how contrived or trivial the coincidence seems, extra new information was needed, on top of homogeneity, in order to construct the Canonical Commutation Relation. Specifically, particular scalings of homogeneity symmetry are needed, in (13) and (14), with $\eta^{-1} = -i$, if we are to construct the algebra of the Canonical Commutation Relation:

$$[\mathbf{k}, \mathbf{x}] = -i\mathbb{1} \quad \text{or} \quad [\mathbf{p}, \mathbf{x}] = -i\hbar\mathbb{1} \tag{15}$$

## Conclusion

The above establishes that the homogeneity symmetry is not a necessary source of unitary information in wave mechanics. And therefore, if the reason given for postulating that quantum theory should be unitary or self-adjoint, is that symmetries in Nature are intrinsically, unavoidably and ontologically unitary, then this one counter-example requires that a different reason be found, or otherwise, the *Postulate* be withdrawn.

This does not mean Quantum Theories are not unitary, because certainly they are; it means that unitarity may not be imposed by the mathematician, for the reason she believes unitarity to be a Fundamental Physical Principle.

## References

1. S357 Course Team; chair: Ray Mackintosh, *Space, time and cosmology unit 3*, no. 1997, The Open University Milton Keynes UK, 1997.