A Generalization of the Einstein-Maxwell Equations II

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Abstract

The proposed modifications of the Einstein-Maxwell equations include: (1) the addition of a scalar term to the electromagnetic side of the equation rather than to the gravitational side, (2) the introduction of a 4-dimensional, nonlinear electromagnetic constitutive tensor and (3) the addition of curvature terms arising from the non-metric components of a general symmetric connection. The scalar term is defined by the condition that a spherically symmetric particle be force-free and mathematically well-behaved everywhere. The constitutive tensor introduces two auxiliary fields which describe the particle structure. The additional curvature terms couple both to particle solutions and to electromagnetic and gravitational wave solutions. This paper corrects and expands earlier work.

1. Introduction

This approach to the construction of a classical unified field theory depends on modifying the Einstein-Maxwell equations in three ways. The first is to move the scalar term, which has been conjectured since the early days of Einstein's cosmological constant, to the electromagnetic side of the equations and to require that it be defined by the condition that a spherically symmetric particle be force-free and mathematically well-behaved everywhere. This simplifies the calculations. The second is to introduce a 4-dimensional electromagnetic constitutive tensor which has two auxiliary field that describe the particle structure. The third is to introduce additional curvature terms on the gravitational side of the equations. These terms arise from the non-metric components of a general symmetric connection and are essential to all of the 4-dimensional solutions. We will begin by looking at Maxwell's equations in a 3-dimensional notation in order to develop a physical understanding of the modifications to the electromagnetic side of the equation. We will then proceed to a 4-dimensional notation and then to a discussion of non-Riemannian geometry leading to the form of the Einstein-Maxwell equations used in this paper. Readers who are familiar with Einstein-Cartan theory and Weyl theory should look especially at Section 4 as an antidote to any preconceived ideas. Sections 5 and 6 gives the particle equations in three and four dimensions. Section 7 gives the equations for the paths (geodesics) and lists the components of the curvature tensor. Specific examples of particle solutions are in Section 8 followed by some graphical results for the normalized charge density and normalized energy density in Section 9. The solutions exhibit a structural transition which may separate leptons from baryons. Section 10 has the discussion of electromagnetic and gravitational waves. The two types of waves are independent of each other and couple only to independent components of the non-Riemannian curvature. The specific examples of particle and wave solutions presented are simple enough to be useful tools in other research. This paper continues the development of work presented by the author at Meetings of the American Physical Society [1], [2]. *Note:* In this paper, 'density' means volume density, not tensor density.

2. Maxwell's Equations in 3-Dimensions

Under certain assumptions, Maxwell's equations can be written in 3-dimensions, using SI units, as:

$$E_i = -\phi_{;i} - \partial_t A_i \qquad B^i = \varepsilon^{ijk} A_{k;j} \tag{2.1a}$$
$$D_i = \epsilon_{ij} E^j - \gamma_{ji} B^j \qquad H_i = \alpha_{ij} B^j + \gamma_{ij} E^j \tag{2.1b}$$
$$\rho = D^i{}_{;i} \qquad j^i = \varepsilon^{ijk} H_{k;j} - \partial_t D^i \tag{2.1c}$$

where ε^{ijk} is the Levi-Civita tensor and α_{ij} is the inverse permeability. In free space, with metric g_{ij} ,

$$\epsilon_{ij} = \epsilon_0 g_{ij} \qquad \alpha_{ij} = \mu_0^{-1} g_{ij} \qquad c^2 \epsilon_0 \mu_0 = 1 \tag{2.2}$$

The following vector-dyadic notation will also be useful:

$$\boldsymbol{D} = \underline{\boldsymbol{\epsilon}} \cdot \boldsymbol{E} - \boldsymbol{B} \cdot \underline{\boldsymbol{\gamma}} \qquad \boldsymbol{H} = \underline{\boldsymbol{\alpha}} \cdot \boldsymbol{B} + \underline{\boldsymbol{\gamma}} \cdot \boldsymbol{E} \tag{2.3}$$

Mathematically, the γ_{ij} terms arise from the fact that, in the 4-dimensional formulation (e.g., E. J. Post [3, pp. 127-134]), the constitutive relations are described by a fourth rank tensor. Physically, they represent a direct coupling between the electric and magnetic fields which traditionally has been thought to be of interest only in material media. The particular form of the coupling used in this paper assumes that there is no optical activity. In this paper, we will show that solutions for which $\boldsymbol{B} = 0$ and $\underline{\boldsymbol{\gamma}} \neq 0$ can be used to represent particles with spin.

We will generalize the traditional definitions of the energy density, the stress tensor and the Poynting vector in two ways. The first is to make the definitions fully symmetric. The second is to introduce a scalar term Q which is motivated by long history of adding scalar fields to General Relativity beginning with Einstein's cosmological constant as the simplest case. In a sense, it can be regarded as simply moving a generalized cosmological term from the gravitational side of the Einstein-Maxwell equations to the electromagnetic side. However, adding a scalar term to the electromagnetic stress-energy tensor turns out to make solving the equations much simpler.

$$En = \tfrac{1}{2}(\alpha_{ij} B^i B^j + \epsilon_{ij} E^i E^j) - Q \tag{2.4a}$$
$$T^{ij} = -\tfrac{1}{2}(E^i D^j + E^j D^i + H^i B^j + H^j B^i) + \tfrac{1}{2} g^{ij} (\alpha_{mn} B^m B^n + \epsilon_{mn} E^m E^n) + g^{ij} Q \tag{2.4b}$$
$$N^i = \tfrac{1}{2} \varepsilon^{ijk} (E_j H_k + c^2 D_j B_k) \tag{2.4c}$$

T^{ij} is defined with the opposite sign from what is usually used in 3-dimensions. It is useful because it lets T^{ij} be the spatial part of $T^{\mu\nu}$, which is defined so that $T^4{}_4 = -En$.

Note that the symmetry in (2.3) ensures that there are no γ_{ij} terms in the energy density, (2.4a). The function Q will be chosen so that the particle solutions are force-free and have finite self-energies. Both classically and quantum mechanically, the problem of force-free particle structures and infinite self-energies was considered by many physicists to be a significant problem until it was eventually swept under the rug by renormalization procedures in quantum field theory. The Born-Infeld [4] approach was one way of dealing with the problem. The approach here opens up a broader realm of mathematically well-behaved solutions which we hope can be used to gain new insights in aspects of black hole theory and quantum field theory that are presently obscured by singularities.

We will define the force density and the power loss density.

$$F_i = -T_i{}^j{}_{;j} - c^{-2} \partial_t N_i \qquad Pwr = -N^i{}_{;i} - \partial_t En \tag{2.5}$$

3. Maxwell's Equations in 4-Dimensions

The electromagnetic fields and the current density are defined by

$$f_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{3.1a}$$
$$p^{\mu\nu} = \tfrac{1}{2} \chi^{\mu\nu\rho\sigma} f_{\rho\sigma} \tag{3.1b}$$
$$j^{\mu} = p^{\mu\nu}{}_{;\nu} \tag{3.1c}$$

where $f_{\mu\nu}$ and $p^{\mu\nu}$ are antisymmetric and the constitutive tensor, $\chi^{\mu\nu\rho\sigma}$, has the symmetries

$$\chi^{\mu\nu\rho\sigma} = -\chi^{\nu\mu\rho\sigma} \qquad \chi^{\mu\nu\rho\sigma} = -\chi^{\mu\nu\sigma\rho} \qquad \chi^{\mu\nu\rho\sigma} = \chi^{\rho\sigma\mu\nu} \tag{3.2}$$

The last of these conditions is the assumption of no optical activity. Post [3, p. 130] argues for additional symmetries which have not been assumed here.

In terms of the 3-dimensional potentials, $A_\mu = c(\boldsymbol{A},\ -\phi)$. We will define the stress-energy tensor and the force density.

$$T^{\mu\nu} = \tfrac{1}{2}(f^{\mu}{}_{\tau}p^{\nu\tau} + f^{\nu}{}_{\tau}p^{\mu\tau}) - g^{\mu\nu}(\tfrac{1}{4}f_{\kappa\tau}p^{\kappa\tau} - Q) \tag{3.3a}$$

$$f_\mu = -T_{\mu}{}^{\nu}{}_{;\nu} \tag{3.3b}$$

As mentioned in Section 2, there is a long history of attempting to add a scalar function to General Relativity. There isn't any mathematical difference in shifting it to the electromagnetic side of the Einstein-Maxwell equations; however, it simplifies the solutions of the equations.

In spherical coordinates in flat space, we adopt the convention that the metric is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\varphi^2 - c^2 dt^2 \tag{3.4}$$

Strictly speaking, this is a pseudo-Riemannian space since the determinant of the metric $g = -c^2 r^4 \sin^2(\theta) < 0$.

4. Non-Riemannian Geometry and the Einstein-Maxwell Equations

Eisenhart [5] shows that the most general asymmetric connection can be written in the form

$$\begin{aligned} &L^{\mu}_{\alpha\beta} = \Omega^{\mu}_{\alpha\beta} + \tilde{\Gamma}^{\mu}_{\alpha\beta} && \tilde{\Gamma}^{\mu}_{\alpha\beta} = a^{\mu}_{\alpha\beta} + \Gamma^{\mu}_{\alpha\beta} && \\ &\Omega^{\mu}_{\alpha\beta} = -\Omega^{\mu}_{\beta\alpha} && a^{\mu}_{\alpha\beta} = a^{\mu}_{\beta\alpha} && \Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha} \end{aligned} \tag{4.1}$$

where $\Omega^{\mu}_{\alpha\beta}$ and $a^{\mu}_{\alpha\beta}$ are tensors and $\Gamma^{\mu}_{\alpha\beta}$ is the metric connection (Christoffel symbols). The curvature tensor for $L^{\mu}_{\alpha\beta}$ can be written as the sum of the curvature tensors for the anti-symmetric part of the connection, $\Omega^{\mu}_{\alpha\beta}$, and the symmetric part of the connection, $\tilde{\Gamma}^{\mu}_{\alpha\beta}$, [5, eq. 5.3].

$$\begin{aligned} &L^{\mu}{}_{\nu\rho\sigma} = \Omega^{\mu}{}_{\nu\rho\sigma} + B^{\mu}{}_{\nu\rho\sigma} && && \\ &L^{\mu}{}_{\nu\rho\sigma} = -L^{\mu}{}_{\nu\sigma\rho} && \Omega^{\mu}{}_{\nu\rho\sigma} = -\Omega^{\mu}{}_{\nu\sigma\rho} && B^{\mu}{}_{\nu\rho\sigma} = -B^{\mu}{}_{\nu\sigma\rho} \end{aligned} \tag{4.2}$$

From [5, eq. 5.5] and (4.1),

$$\begin{aligned} \Omega^{\mu}{}_{\nu\rho\sigma} &= \Omega^{\mu}_{\nu\sigma|\rho} - \Omega^{\mu}_{\nu\rho|\sigma} + \Omega^{\mu}_{\alpha\sigma}\Omega^{\alpha}_{\nu\rho} - \Omega^{\mu}_{\alpha\rho}\Omega^{\alpha}_{\nu\sigma} - 2\Omega^{\mu}_{\nu\alpha}\Omega^{\alpha}_{\rho\sigma} \\ &= \Omega^{\mu}_{\nu\sigma;\rho} - \Omega^{\mu}_{\nu\rho;\sigma} + \Omega^{\mu}_{\alpha\rho}\Omega^{\alpha}_{\nu\sigma} - \Omega^{\mu}_{\alpha\sigma}\Omega^{\alpha}_{\nu\rho} + 2\Omega^{\mu}_{\alpha\rho}a^{\alpha}_{\nu\sigma} - 2\Omega^{\mu}_{\alpha\sigma}a^{\alpha}_{\nu\rho} \end{aligned} \tag{4.3}$$

From [5, eq. 5.15],

$$B^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + a^{\mu}_{\nu\sigma;\rho} - a^{\mu}_{\nu\rho;\sigma} + a^{\alpha}_{\nu\sigma}a^{\mu}_{\alpha\rho} - a^{\alpha}_{\nu\rho}a^{\mu}_{\alpha\sigma} \tag{4.4}$$

A solidus ("|") denotes covariant differentiation with respect to the asymmetric connection $L^{\mu}_{\alpha\beta}$, a colon denotes covariant differentiation with respect to the general symmetric connection $\tilde{\Gamma}^{\mu}_{\alpha\beta}$, a semicolon denotes covariant differentiation with respect to the metric connection $\Gamma^{\mu}_{\alpha\beta}$ and a comma will denote partial differentiation with respect to the coordinates. (This notation is somewhat different from that used by Eisenhart. He uses the Christoffel symbols for the metric connection and $\Gamma^{\mu}_{\alpha\beta}$ for the general symmetric connection. More importantly, he usually uses a comma to denote covariant differentiation with respect to the general symmetric connection.) $R^{\mu}{}_{\nu\rho\sigma}$ is the Riemann curvature tensor for the metric $g_{\mu\nu}$. Covariant differentiation with respect to the metric is more convenient than covariant differentiation with respect to the asymmetric connection for at least two reasons. First is the fact that

$$\begin{aligned} g_{\mu\nu|\tau} &= -g_{\alpha\nu}(\Omega^{\alpha}_{\mu\tau} + a^{\alpha}_{\mu\tau}) - g_{\mu\alpha}(\Omega^{\alpha}_{\nu\tau} + a^{\alpha}_{\nu\tau}) \neq 0 \\ g^{\mu\nu}{}_{|\tau} &= g^{\alpha\nu}(\Omega^{\mu}_{\alpha\tau} + a^{\mu}_{\alpha\tau}) + g^{\mu\alpha}(\Omega^{\nu}_{\alpha\tau} + a^{\nu}_{\alpha\tau}) \neq 0 \end{aligned} \tag{4.5}$$

Second, the commuting of the covariant derivatives is more complicated.

$$w_{|\rho\sigma} - w_{|\sigma\rho} = -2w_{|\alpha}\Omega^{\alpha}_{\rho\sigma} \tag{4.6a}$$
[5, eq. 4.1]

$$w^{\mu}{}_{|\rho\sigma} - w^{\mu}{}_{|\sigma\rho} = -w^{\alpha}L^{\mu}{}_{\alpha\rho\sigma} - 2w^{\mu}{}_{|\alpha}\Omega^{\alpha}_{\rho\sigma} \tag{4.6b}$$
[5, eq. 4.2]

$$w_{\mu|\rho\sigma} - w_{\mu|\sigma\rho} = w_{\alpha}L^{\alpha}{}_{\mu\rho\sigma} - 2w_{\mu|\alpha}\Omega^{\alpha}_{\rho\sigma} \tag{4.6c}$$
[5, eq. 4.3]

$$w_{\mu\nu|\rho\sigma} - w_{\mu\nu|\sigma\rho} = w_{\alpha\nu}L^{\alpha}{}_{\mu\rho\sigma} + w_{\mu\alpha}L^{\alpha}{}_{\nu\rho\sigma} - 2w_{\mu\nu|\alpha}\Omega^{\alpha}_{\rho\sigma} \tag{4.6d}$$
[5, eq. 4.4]

For these reasons, the equations are expressed in terms of covariant differentiation with respect to the metric connection, $\Gamma^{\mu}_{\alpha\beta}$. Define

$$\Omega_{\nu\sigma} = \Omega^{\mu}{}_{\nu\mu\sigma} = \Omega^{\mu}{}_{\nu\sigma;\mu} - \Omega^{\mu}_{\nu\mu;\sigma} + \Omega^{\mu}_{\alpha\mu}\Omega^{\alpha}_{\nu\sigma} - \Omega^{\mu}_{\alpha\sigma}\Omega^{\alpha}_{\nu\mu} + 2\Omega^{\mu}_{\alpha\mu}a^{\alpha}_{\nu\sigma} - 2\Omega^{\mu}_{\alpha\sigma}a^{\alpha}_{\nu\mu} \tag{4.7a}$$

$$B_{\nu\sigma} = B^{\mu}{}_{\nu\mu\sigma} = R^{\mu}{}_{\nu\mu\sigma} + a^{\mu}_{\nu\sigma;\mu} - a^{\mu}_{\nu\mu;\sigma} + a^{\alpha}_{\nu\sigma}a^{\mu}_{\alpha\mu} - a^{\alpha}_{\nu\mu}a^{\mu}_{\alpha\sigma} \tag{4.7b}$$

Define symmetric and antisymmetric parts in the following way:

$$S_{\Omega\nu\sigma} = \tfrac{1}{2}(\Omega_{\nu\sigma} + \Omega_{\sigma\nu}) = -\tfrac{1}{2}(\Omega^{\mu}_{\nu\mu;\sigma} + \Omega^{\mu}_{\sigma\mu;\nu}) + \Omega^{\mu}_{\alpha\sigma}\Omega^{\alpha}_{\mu\nu} + 2\Omega^{\mu}_{\alpha\mu}a^{\alpha}_{\nu\sigma} - \Omega^{\mu}_{\alpha\sigma}a^{\alpha}_{\mu\nu} - \Omega^{\mu}_{\alpha\nu}a^{\alpha}_{\mu\sigma} \tag{4.8a}$$

$$A_{\Omega\nu\sigma} = \tfrac{1}{2}(\Omega_{\nu\sigma} - \Omega_{\sigma\nu}) = \Omega^{\mu}_{\nu\sigma;\mu} - \tfrac{1}{2}(\Omega^{\mu}_{\nu\mu;\sigma} - \Omega^{\mu}_{\sigma\mu;\nu}) + \Omega^{\mu}_{\alpha\mu}\Omega^{\alpha}_{\nu\sigma} - \Omega^{\mu}_{\alpha\sigma}a^{\alpha}_{\mu\nu} + \Omega^{\mu}_{\alpha\nu}a^{\alpha}_{\mu\sigma} \tag{4.8b}$$

$$S_{B\nu\sigma} = \tfrac{1}{2}(B_{\nu\sigma} + B_{\sigma\nu}) - R_{\nu\sigma} = a^{\mu}_{\nu\sigma;\mu} - \tfrac{1}{2}(a^{\mu}_{\nu\mu;\sigma} + a^{\mu}_{\sigma\mu;\nu}) + a^{\alpha}_{\nu\sigma}a^{\mu}_{\alpha\mu} - a^{\alpha}_{\nu\mu}a^{\mu}_{\alpha\sigma} \tag{4.8c}$$

$$A_{B\nu\sigma} = \tfrac{1}{2}(B_{\nu\sigma} - B_{\sigma\nu}) = -\tfrac{1}{2}(a^{\mu}_{\nu\mu;\sigma} - a^{\mu}_{\sigma\mu;\nu}) \tag{4.8d}$$

The spin is described by the non-Riemannian part of the connection. In this paper, we have assumed

$$\Omega^{\mu}_{\alpha\beta} = 0 \qquad a^{\mu}_{\alpha\mu} = 0 \qquad g^{\mu\nu}a^{\sigma}_{\mu\nu} = 0 \tag{4.9}$$

The first two of these constraints are sufficient to give

$$S_{\Omega\mu\nu} = 0 \qquad A_{\Omega\mu\nu} = 0 \qquad S_{B\nu\sigma} = a^{\mu}_{\nu\sigma;\mu} - a^{\alpha}_{\mu\nu}a^{\mu}_{\alpha\sigma} \qquad A_{B\nu\sigma} = 0 \tag{4.10}$$

Hence we will write the generalized form of the Einstein-Maxwell equations as

$$G_{\mu\nu} + S_{B\mu\nu} = 8\pi G c^{-4}T_{\mu\nu} \qquad G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R \tag{4.11}$$

where G is Newton's gravitational constant.

The condition $a^{\mu}_{\alpha\mu} = 0$ distinguishes this connection from Weyl's symmetric connection [5, §30]. In Eisenhart's terminology, it also forces the paths of $\tilde{\Gamma}^{\mu}_{\alpha\beta}$ to be different from the geodesics of $\Gamma^{\mu}_{\alpha\beta}$ [5, §7, §12, §22]. In every coordinate system,

$$a^{\mu}_{\alpha\mu} = a^{1}_{\alpha 1} + a^{2}_{\alpha 2} + a^{3}_{\alpha 3} + a^{4}_{\alpha 4} = 0 \tag{4.12}$$

Thus, for example, we can have $a^{1}_{24} \neq 0$ and $a^{4}_{21} = 0$ in a particular frame if needed. The solutions presented in this paper are such that for each one there exists a coordinate system in which

$$a^{1}_{\alpha 1} = a^{2}_{\alpha 2} = a^{3}_{\alpha 3} = a^{4}_{\alpha 4} = 0 \tag{4.13}$$

The constraint $g^{\mu\nu}a^{\sigma}_{\mu\nu} = 0$ is new. It comes from considering the divergence of an arbitrary vector V^{μ}. From (4.9)

$$V^{\mu}{}_{:\mu} = V^{\mu}{}_{;\mu} + V^{\nu}a^{\mu}_{\nu\mu} = V^{\mu}{}_{;\mu} \tag{4.14a}$$

$$g^{\mu\nu}V_{\mu:\nu} = g^{\mu\mu}V_{\mu;\nu} - g^{\mu\mu}V_{\sigma}a^{\sigma}_{\mu\nu} = g^{\mu\nu}V_{\mu;\nu} \tag{4.14b}$$

Thus the non-Riemannian divergence of any vector reduces to the Riemannian divergence and, from (4.5), the non-Riemannian divergence of the metric tensor is zero. This constraint further restricts the forms of the particle and wave spin connections given in §6 and §10 below.

The approach used in this paper will seem strange to readers who are more accustomed to Einstein-Cartan theory in which the assumption is that $a^{\mu}_{\alpha\beta} = 0$ and $\Omega^{\mu}_{\alpha\beta} \neq 0$. For a review of Einstein-Cartan theory, see F. W. Hehl *et al* [6]. One advantage of this approach is the ease of constructing explicit particle and wave solutions.

5. Particle Equations in 3-Dimensions

In spherical coordinates (r, θ, φ) , let

$$\boldsymbol{E} = f_e(r)\boldsymbol{e}_r = -\boldsymbol{e}_r\phi'(r) \tag{5.1a}$$
$$\boldsymbol{A} = 0 \tag{5.1b}$$
$$\underline{\boldsymbol{\alpha}} = c^2\underline{\boldsymbol{\epsilon}} = c^2\epsilon_0 f_\epsilon(r)(\boldsymbol{e}_r\boldsymbol{e}_r + \boldsymbol{e}_\theta\boldsymbol{e}_\theta + \boldsymbol{e}_\varphi\boldsymbol{e}_\varphi) \tag{5.1c}$$
$$\underline{\boldsymbol{\gamma}} = h(r)[(2\boldsymbol{e}_r\boldsymbol{e}_r - \boldsymbol{e}_\theta\boldsymbol{e}_\theta - \boldsymbol{e}_\varphi\boldsymbol{e}_\varphi)\cos(\theta) + (\boldsymbol{e}_r\boldsymbol{e}_\theta + \boldsymbol{e}_\theta\boldsymbol{e}_r)\sin(\theta))] \tag{5.1d}$$

where the form of $\underline{\boldsymbol{\gamma}}$ has been chosen by trial and error so that in weak external fields there are no singularities in the various volume integrals for force, momentum, etc. Then

$$\boldsymbol{D} = \epsilon_0 f_\epsilon(r) f_e(r)\boldsymbol{e}_r \tag{5.2a}$$
$$\rho(r) = \epsilon_0 r^{-2}\partial_r[r^2 f_e(r) f_\epsilon(r)] \tag{5.2b}$$
$$\boldsymbol{H} = f_e(r)h(r)[2\cos(\theta)\boldsymbol{e}_r + \sin(\theta)\boldsymbol{e}_\theta] \tag{5.2c}$$
$$\boldsymbol{j} = r^{-3}\partial_r[r^3 f_e(r)h(r)]\sin(\theta)\boldsymbol{e}_\varphi \tag{5.2d}$$
$$Q(r) = \tfrac{1}{2}\epsilon_0 f_e^2(r) f_\epsilon(r) - 2\epsilon_0\int_r^\infty dr'\,(r')^{-1} f_e^2(r') f_\epsilon(r') \tag{5.2e}$$
$$\underline{\boldsymbol{T}} = \tfrac{1}{2}\epsilon_0 f_e^2(r) f_\epsilon(r)[-\boldsymbol{e}_r\boldsymbol{e}_r + \boldsymbol{e}_\theta\boldsymbol{e}_\theta + \boldsymbol{e}_\varphi\boldsymbol{e}_\varphi] + Q(r)[\boldsymbol{e}_r\boldsymbol{e}_r + \boldsymbol{e}_\theta\boldsymbol{e}_\theta + \boldsymbol{e}_\varphi\boldsymbol{e}_\varphi] \tag{5.2f}$$
$$En(r) = \tfrac{1}{2}\epsilon_0 f_e^2(r) f_\epsilon(r) - Q(r) = 2\epsilon_0\int_r^\infty dr'\,(r')^{-1} f_e^2(r') f_\epsilon(r') \tag{5.2g}$$
$$\boldsymbol{N} = \tfrac{1}{2}h(r) f_e^2(r)\sin(\theta)\boldsymbol{e}_\varphi \tag{5.2h}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{H} = 2r^{-3}\partial_r[r^3 f_e(r)h(r)]\cos(\theta) \tag{5.2i}$$

For finite, continuously differentiable functions, these solutions are force free and radiationless. As $r \to 0$, we must have $f_e(r) \to 0$ and $h(r) \to 0$ fast enough that there are no singularities and no directional dependence on θ or ϕ. For charged particles with spin, we must also have

$$\lim_{r\to\infty} f_e(r) = q(4\pi\epsilon_0 r^2)^{-1} \tag{5.3a}$$
$$\lim_{r\to\infty} f_\epsilon(r) = 1 \tag{5.3b}$$
$$\lim_{r\to\infty} f_e(r)h(r) = \gamma(4\pi r^3)^{-1} \tag{5.3c}$$

In order to minimize any disagreement with experimental results in the far field, we will require that the limits in (5.3) be approached exponentially rather than polynomially. Note that the restrictions on $f_\epsilon(r)$ are much different than in standard Born-Infeld theory [4]. Note also that the expression for $Q(r)$ in (5.2e) is an integral expression in the electromagnetic field rather than a local expression. The limits of the integral have been chosen to insure the correct asymptotic behavior as $r \to \infty$. We will show later that $Q(r)$ is local in terms of the electromagnetic field and the curved metric. Another point is that even though $\boldsymbol{\nabla}\cdot\boldsymbol{H} \neq 0$, the $\cos(\theta)$ factor prevents the existence of any magnetic monopoles q_m in this theory.

$$\begin{aligned} q_m &= \int_0^\infty dr\, r^2 \int_0^\pi d\theta\,\sin(\theta)\int_0^{2\pi} d\varphi\;\boldsymbol{\nabla}\cdot\boldsymbol{H} \\ &= 0 \end{aligned} \tag{5.4}$$

We can define a pseudo magnetic monopole $q_{\rm H}$ over the upper half of the sphere.

$$q_{\rm H} = \int_0^\infty dr\, r^2 \int_0^{\pi/2} d\theta\,\sin(\theta)\int_0^{2\pi} d\varphi\;\boldsymbol{\nabla}\cdot\boldsymbol{H} \tag{5.5}$$

This is balanced by an equal and opposite pseudo magnetic monopole $-q_{\rm H}$ over the lower half of the sphere.

The rest mass

$$m_0 = c^{-2}\int_0^\infty dr\, r^2 \int_0^\pi d\theta\, \sin(\theta) \int_0^{2\pi} d\varphi\; En(r)$$
$$= 8\pi\epsilon_0 c^{-2}\int_0^\infty dr\, r^2 \int_r^\infty dr'\, (r')^{-1} f_e^2(r') f_\epsilon(r') \tag{5.6}$$

Since

$$\begin{aligned} \boldsymbol{e}_r &= \sin(\theta)\cos(\varphi)\boldsymbol{e}_x + \sin(\theta)\sin(\varphi)\boldsymbol{e}_y + \cos(\theta)\boldsymbol{e}_z \\ \boldsymbol{e}_\theta &= \cos(\theta)\cos(\varphi)\boldsymbol{e}_x + \cos(\theta)\sin(\varphi)\boldsymbol{e}_y - \sin(\theta)\boldsymbol{e}_z \\ \boldsymbol{e}_\varphi &= -\sin(\varphi)\boldsymbol{e}_x + \cos(\varphi)\boldsymbol{e}_y \end{aligned} \tag{5.7}$$

the total momentum and angular momentum are given by

$$\begin{aligned} \boldsymbol{P}_{\mathrm{T}} &= c^{-2}\int_0^\infty dr\, r^2 \int_0^\pi d\theta\, \sin(\theta) \int_0^{2\pi} d\varphi\; \boldsymbol{N} \\ &= 0 \end{aligned} \tag{5.8a}$$

$$\begin{aligned} \boldsymbol{J}_{\mathrm{T}} &= c^{-2}\int_0^\infty dr\, r^2 \int_0^\pi d\theta\, \sin(\theta) \int_0^{2\pi} d\varphi\; \boldsymbol{r}\times\boldsymbol{N} \\ &= \tfrac{4}{3}\pi c^{-2}\int_0^\infty dr\, r^3 h(r) f_e^2(r)\boldsymbol{e}_z \end{aligned} \tag{5.8b}$$

The total charge, total current and total angular moment of the current are defined by

$$\begin{aligned} q_{\mathrm{T}} &= \int_0^\infty dr\, r^2 \int_0^\pi d\theta\, \sin(\theta) \int_0^{2\pi} d\varphi\, \rho(r) \\ &= 4\pi\epsilon_0\; r^2 f_\epsilon(r) f_e(r)\big|_0^\infty \end{aligned} \tag{5.9a}$$

$$\begin{aligned} \boldsymbol{j}_{\mathrm{T}} &= \int_0^\infty dr\, r^2 \int_0^\pi d\theta\, \sin(\theta) \int_0^{2\pi} d\varphi\, \boldsymbol{j} \\ &= 0 \end{aligned} \tag{5.9b}$$

$$\begin{aligned} \boldsymbol{M}_{\mathrm{T}} &= \int_0^\infty dr\, r^2 \int_0^\pi d\theta\, \sin(\theta) \int_0^{2\pi} d\varphi\, \boldsymbol{r}\times\boldsymbol{j} \\ &= \tfrac{8}{3}\pi\boldsymbol{e}_z\; r^3 h(r) f_e(r)\big|_{r=0}^\infty \end{aligned} \tag{5.9c}$$

Note that for a neutral particle, $q_{\mathrm{T}} = 0$ which puts a constraint on the asymptotic behavior of $f_e(r)$.

If there are external fields with potentials

$$\begin{aligned} \phi_{\mathrm{ext}} &= -(E_{0x}x + E_{0y}y + E_{0z}z) \\ \boldsymbol{A}_{\mathrm{ext}} &= \tfrac{1}{2}[(B_{0y}z - B_{0z}y)\boldsymbol{e_x} + (B_{0z}x - B_{0x}z)\boldsymbol{e_y} + (B_{0x}y - B_{0y}x)\boldsymbol{e_z}] \end{aligned} \tag{5.10}$$

then we have the constant fields

$$\boldsymbol{E}_0 = E_{0x}\boldsymbol{e_x} + E_{0y}\boldsymbol{e_y} + E_{0z}\boldsymbol{e_z} \qquad \boldsymbol{B}_0 = B_{0x}\boldsymbol{e_x} + B_{0y}\boldsymbol{e_y} + B_{0z}\boldsymbol{e_z} \tag{5.11}$$

If we assume that the external fields do not, to a first approximation, modify $f_\epsilon(r)$, $h(r)$ and $Q(r)$ and if the accelerations are low so that radiation reaction effects can be ignored, then the total force and the total torque are

$$\begin{aligned}\boldsymbol{F}_{\mathrm{T}} &= \int_0^\infty dr\, r^2 \int_0^\pi d\theta\, \sin(\theta) \int_0^{2\pi} d\varphi\, \boldsymbol{F} \\ &= 4\pi\epsilon_0 r^2 f_\epsilon(r) f_e(r) \,|_{r=0}^{\infty} \boldsymbol{E}_0 \\ &= q_{\mathrm{T}} \boldsymbol{E}_0 \\ &= q\boldsymbol{E}_0, \text{ if } q_{\mathrm{T}} \neq 0\end{aligned} \tag{5.12a}$$

$$\begin{aligned}\boldsymbol{W}_{\mathrm{T}} &= \int_0^\infty dr\, r^2 \int_0^\pi d\theta\, \sin(\theta) \int_0^{2\pi} d\varphi\, \boldsymbol{r} \times \boldsymbol{F} \\ &= 2\pi r^3 h(r) f_e(r)\big|_{r=0}^{\infty} \boldsymbol{e}_z \times \boldsymbol{B}_0 \\ &= \tfrac{3}{4} \boldsymbol{M}_{\mathrm{T}} \times \boldsymbol{B}_0 \\ &= \tfrac{1}{2}\gamma \boldsymbol{e}_z \times \boldsymbol{B}_0, \text{ if } \boldsymbol{M}_{\mathrm{T}} \neq 0\end{aligned} \tag{5.12b}$$

The factor of $\frac{1}{2}$ in $\boldsymbol{W}_{\mathrm{T}}$ distinguishes this result from the normal magnetic dipole, $\boldsymbol{W}_{\mathrm{T}} = \mu_{\mathrm{m}} \boldsymbol{e}_z \times \boldsymbol{B}$. Thus the numerical values for γ are related to the numerical values reported for μ_{m} by

$$\gamma = 2\mu_{\mathrm{m}} \tag{5.13}$$

This corresponds to the quantum spin factor $g_s = 2$.

We can define an effective rest mass energy for the particle by subtracting the unperturbed energy density of the external field. In this case, referring back to the general definition in (2.4a),

$$\begin{aligned}m_{\mathrm{eff}} c^2 &= \int_0^\infty dr\, r^2 \int_0^\pi d\theta\, \sin(\theta) \int_0^{2\pi} d\varphi\, [En - \tfrac{1}{2}\epsilon_0 (B_0^2 c^2 + E_0^2)] \\ &= m_0 c^2 + 2\pi\epsilon_0 (B_0^2 c^2 + E_0^2) \int_0^\infty dr\, r^2 [f_\epsilon(r) - 1]\end{aligned} \tag{5.14}$$

Even though the particle is at rest, we can define an effective total field momentum and an effective total angular momentum by subtracting the unperturbed Poynting vector at infinity. In this case,

$$\boldsymbol{N}_0 = c^2 \epsilon_0 [q_{\mathrm{T}} (4\pi\epsilon_0 r^2)^{-1} \boldsymbol{e}_r + \boldsymbol{E}_0] \times \boldsymbol{B}_0 \tag{5.15a}$$

$$\begin{aligned}\boldsymbol{P}_{\mathrm{T\ eff}} &= c^{-2} \int_0^\infty dr\, r^2 \int_0^\pi d\theta\ \sin(\theta) \int_0^{2\pi} d\varphi\ (\boldsymbol{N} - \boldsymbol{N}_0) \\ &= 4\pi\epsilon_0 \boldsymbol{E}_0 \times \boldsymbol{B}_0 \int_0^\infty dr\, r^2 [f_\epsilon(r) - 1]\end{aligned} \tag{5.15b}$$

$$\begin{aligned}\boldsymbol{J}_{\mathrm{T\ eff}} &= c^{-2} \int_0^\infty dr\, r^2 \int_0^\pi d\theta\, \sin(\theta) \int_0^{2\pi} d\varphi\ \boldsymbol{r} \times (\boldsymbol{N} - \boldsymbol{N}_0) \\ &= \boldsymbol{J}_{\mathrm{T}} + \tfrac{2}{3} \boldsymbol{B}_0 \int_0^\infty dr\, r [q_{\mathrm{T}} - 4\pi\epsilon_0 r^2 f_\epsilon(r) f_e(r)]\end{aligned} \tag{5.15c}$$

We can define the angular momentum about another point P as

$$\boldsymbol{J}_P = \boldsymbol{J}_{\mathrm{T\ eff}} + \boldsymbol{r}_P \times \boldsymbol{P}_{\mathrm{T\ eff}} \tag{5.16}$$

where $\boldsymbol{r}_P$ is the radius vector from the point P to the center of the particle. In standard quantum theory, this is valid for orbital angular momenta, but not for spin. It is possible to construct solutions for which $m_{\mathrm{eff}} = m_0$, $\boldsymbol{P}_{\mathrm{T\ eff}} = 0$ and $\boldsymbol{J}_{\mathrm{T\ eff}} = \boldsymbol{J}_{\mathrm{T}}$ by adding additional terms to $f_\epsilon(r)$. However in the limit of weak external fields, the effect is negligibly small for the elementary particles and the basic structure is better shown if we do not add any extra terms. The total effective current and the total effective angular moment of the current are

$$\boldsymbol{j}_{\mathrm{T\ eff}} = 0 \tag{5.17a}$$

$$\boldsymbol{M}_{\mathrm{T\ eff}} = \boldsymbol{M}_{\mathrm{T}} - \tfrac{8}{3}\pi\epsilon_0 c^2 \boldsymbol{B}_0 \int_0^\infty dr\, r^3 f'_\epsilon(r) \tag{5.17b}$$

In classical theory, every static magnetic field is associated with a current of moving charges. In quantum theory, static magnetic fields are associated either with a current of moving charges or with a fixed array of particles that have spin. In this theory, $\boldsymbol{B} = 0$ for spin fields, but not for moving charges. This would seem to be one reason why it has been difficult to explore the mathematical transition between quantum theory and classical theory.

6. Particle Equations in 4-Dimensions

In the rest frame of a particle, let the metric be given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = f_g^{-1}(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 - c^2 f_g(r) dt^2 \tag{6.1}$$

and, in accordance with (4.9), let the only non-zero component of $a^\mu_{\nu\sigma}$ be

$$a^1_{43} = -\zeta(r) \sin^2(\theta) \tag{6.2}$$

If

$$A_\mu = (0{,}0{,}0, -c\phi(r)) \qquad \phi(r) = -\int dr\, f_e(r) \tag{6.3}$$

and if the metric and non-metric components of the constitutive tensor are specified by

$$\begin{aligned}
\chi_{\mu\nu\rho\sigma} &= \epsilon_0 f_\epsilon(r)(g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) \\
\chi_{3241} &= -2r^2 h(r) \sin(\theta) \cos(\theta) \\
\chi_{3242} &= -r^2 f_g^2(r) \chi_{3141} = -r^3 h(r) f_g(r) \sin^2(\theta) \\
\chi_{2143} &= -\chi_{3142} = r^2 h(r) \sin(\theta) \cos(\theta)
\end{aligned} \tag{6.4}$$

then the non-zero components of $T_{\mu\nu}$, $G_{\mu\nu}$ and $S_{B\mu\nu}$ are

$$T_{44} = -c^2 f_g^2(r) T_{11} = 2c^2 \epsilon_0 f_g(r) \int_r^\infty dr'\, (r')^{-1} f_e^2(r') f_\epsilon(r') \tag{6.5a}$$

$$T_{33} = \sin^2(\theta) T_{22} = \epsilon_0 r^2 [f_e^2(r) f_\epsilon(r) - 2 \int_r^\infty dr'\, (r')^{-1} f_e^2(r') f_\epsilon(r')] \sin^2(\theta) \tag{6.5b}$$

$$T_{34} = -\tfrac{1}{2} r h(r) f_e^2(r) \sin^2(\theta) \tag{6.5c}$$

$$G_{44} = -c^2 f_g^2(r) G_{11} = -c^2 r^{-2} f_g(r) [-1 + f_g(r) + r f_g'(r)] \tag{6.5d}$$

$$G_{33} = \sin^2(\theta) G_{22} = [r f_g'(r) + \tfrac{1}{2} r^2 f_g''(r)] \sin^2(\theta) \tag{6.5e}$$

$$S_{B34} = \sin^2(\theta) \{\zeta(r)[2 f_g(r) - r f_g'(r)][2r f_g(r)]^{-1} + \zeta'(r)\} \tag{6.5f}$$

Equations (4.11) reduce to

$$-1 + f_g(r) + r f_g'(r) = -16\pi G c^{-4} \epsilon_0 r^2 \int_r^\infty dr'\, (r')^{-1} f_e^2(r') f_\epsilon(r') \tag{6.6a}$$

$$r f_g'(r) + \tfrac{1}{2} r^2 f_g''(r) = 8\pi G c^{-4} \epsilon_0 r^2 [f_e^2(r) f_\epsilon(r) - 2 \int_r^\infty dr'\, (r')^{-1} f_e^2(r') f_\epsilon(r')] \tag{6.6b}$$

$$\zeta(r)[2 f_g(r) - r f_g'(r)][2r f_g(r)]^{-1} + \zeta'(r) = 4\pi G c^{-4} r h(r) f_e^2(r) \tag{6.6c}$$

Integrating (6.6b) and substituting into (6.6a) gives

$$f_g(r) = 1 - \frac{16\pi G \epsilon_0}{c^4 r} \int_0^r dr'\, (r')^2 \int_{r'}^\infty dr''\, (r'')^{-1} f_e^2(r'') f_\epsilon(r'') \tag{6.7}$$

Comparison with the Schwarzschild metric, for which $f_g(r) = 1 - 2Gm_0 c^{-2} r^{-1}$, shows that

$$m_0 = 8\pi \epsilon_0 c^{-2} \int_0^\infty dr\, r^2 \int_r^\infty dr'\, (r')^{-1} f_e^2(r') f_\epsilon(r') \tag{6.8}$$

which agrees with (5.6). The entire rest mass is electromagnetic. For a charged particle with the asymptotic form (5.3), we can use

$$\int_0^r dr' = \int_0^\infty dr' - \int_r^\infty dr' \tag{6.9}$$

to show that

$$\lim_{r\to\infty} f_g(r) = 1 - \frac{2Gm_0}{c^2 r} + \frac{G q_{\mathrm{T}}^2}{4\pi\epsilon_0 c^4 r^2} + \cdots \tag{6.10}$$

where $q_{\mathrm{T}} = q$ for a charged particle and $q_{\mathrm{T}} = 0$ for a neutral particle. The first three terms agree with the Reissner-Nordstöm metric. We will construct solutions such that the higher-order correction terms decrease at least as fast as $\exp[-(r/r_0)^3]$ where r_0 is determined by the size of the particle. This should ensure agreement with experimental results in the far field.

Note that (5.2e) and (6.6a) give

$$Q(r) = \tfrac{1}{2}\epsilon_0 f_e^2(r) f_\epsilon(r) + c^4 (8\pi G r^2)^{-1} [f_g(r) + r f_g'(r) - 1] \tag{6.11}$$

Thus $Q(r)$ can be expressed as the difference between the traditional form of Maxwell's energy density and Einstein's gravitational energy density. It is a local function in terms of the electromagnetic field and the curved metric. The advantage of (5.2e) is that it is explicitly derived from the condition $T^{\mu\nu}{}_{;\nu} = 0$.

Finally, (6.6c) gives

$$\zeta(r) = \frac{4\pi G}{c^4 r} [f_g(r)]^{1/2} \int_0^r dr' (r')^2\, h(r') f_e^2(r') [f_g(r')]^{-1/2} \tag{6.12}$$

We can use the asymptotic form (5.3) together with (6.9) and (6.10) to show that

$$\lim_{r\to\infty} \zeta(r) = \frac{4\pi G}{c^4 r} \int_0^\infty dr' (r')^2\, h(r') f_e^2(r') + \cdots \tag{6.13}$$

if the space is sufficiently flat that $f_g(r) \approx 1$ throughout the entire region of integration. Detailed calculations show that

$$8\pi G c^{-4} T^{\mu\nu}{}_{;\nu} = S_B{}^{\mu\nu}{}_{;\nu} = 0 \tag{6.14}$$

7. Paths and Curvature Tensor

Eisenhart [5] reserves the term "geodesics" for Riemannian spaces and uses the term "paths" for non-Riemannian spaces. The equations for the paths [5, § 22], in terms of an affine parameter, ξ , are

$$\frac{d^2 x^\mu}{d\xi^2} + \frac{dx^\alpha}{d\xi}\frac{dx^\beta}{d\xi}\tilde{\Gamma}^\mu_{\alpha\beta} = \left(\frac{dx^\mu}{d\xi}\right)_{;\nu} \frac{dx^\nu}{d\xi} = 0 \tag{7.1}$$

In detail,

$$0 = \frac{d^2 r}{d\xi^2} + \frac{1}{2}\frac{df_g(r)}{dr}\left[c^2 f_g(r)\left(\frac{dt}{d\xi}\right)^2 - \frac{1}{f_g(r)}\left(\frac{dr}{d\xi}\right)^2\right] - r f_g(r)\left[\left(\frac{d\theta}{d\xi}\right)^2 + \sin^2(\theta)\left(\frac{d\varphi}{d\xi}\right)^2\right] - 2\zeta(r)\sin^2(\theta)\frac{d\varphi}{d\xi}\frac{dt}{d\xi} \tag{7.2a}$$

$$0 = \frac{d^2\theta}{d\xi^2} + \frac{2}{r}\frac{d\theta}{d\xi}\frac{dr}{d\xi} - \cos(\theta)\sin(\theta)\left(\frac{d\varphi}{d\xi}\right)^2 \tag{7.2b}$$

$$0 = \frac{d^2\varphi}{d\xi^2} + \frac{2\cos(\theta)}{\sin(\theta)}\frac{d\varphi}{d\xi}\frac{d\theta}{d\xi} + \frac{2}{r}\frac{d\varphi}{d\xi}\frac{dr}{d\xi} \tag{7.2c}$$

$$0 = \frac{d^2 t}{d\xi^2} + \frac{1}{f_g(r)}\frac{df_g(r)}{dr}\frac{dt}{d\xi}\frac{dr}{d\xi} \tag{7.2d}$$

One solution is given by

$$\frac{d\theta}{d\xi}=\frac{d\varphi}{d\xi}=\frac{dt}{d\xi}=0 \qquad 0=\frac{d^2r}{d\xi^2}-\frac{1}{2f_g(r)}\frac{df_g(r)}{dr}\left(\frac{dr}{d\xi}\right)^2 \qquad \xi=C\int_0^r dr'[f_g(r')]^{-1/2} \tag{7.3}$$

where C is a constant. In order for ξ to be real, $f_g(r)\geq 0$. Another solution exists for which

$$\theta=\frac{\pi}{2} \qquad \frac{dr}{d\xi}=\frac{d\theta}{d\xi}=0 \tag{7.4}$$

In this case, (7.2) reduces to

$$0=\frac{c^2}{2}\frac{df_g(r)}{dr}f_g(r)\left(\frac{dt}{d\xi}\right)^2-rf_g(r)\left(\frac{d\varphi}{d\xi}\right)^2-2\zeta(r)\frac{d\varphi}{d\xi}\frac{dt}{d\xi} \qquad 0=0 \qquad 0=\frac{d^2\varphi}{d\xi^2} \qquad 0=\frac{d^2t}{d\xi^2} \tag{7.5}$$

Thus, we can set

$$\varphi=\omega(r)\xi \quad t=\xi \quad \varphi=\omega(r)t \quad v_\varphi=r\,\omega(r) \quad \omega(r)=\left[-\zeta(r)\pm\sqrt{\zeta^2(r)+\tfrac{1}{2}rc^2f_g^2(r)\frac{df_g(r)}{dr}}\right]\frac{1}{rf_g(r)} \tag{7.6}$$

where $\omega(r)$ is the angular velocity of a hypothetical test particle moving on the path with velocity v_φ. In the far field, (6.10), (6.14) and (7.6) give

$$\lim_{r\to\infty}\omega(r)=\pm\sqrt{Gm_0r^{-3}} \tag{7.10}$$

These particular geodesics are valid only in regions in which $\omega(r)$ is real. Perhaps the minimum value of r for which (7.10) is valid is a measure of the size of the particle.

For particles, the components of the curvature tensor $B_{\mu\nu\rho\sigma}$ are

$$\begin{aligned}
&B_{4343}=\tfrac{1}{2}c^2rf_g(r)f_g'(r)\sin^2(\theta) && B_{4241}=0 && B_{4121}=0\\
&B_{4342}=0 && B_{4232}=-r\zeta(r)\sin^2(\theta) && B_{3232}=r^2[1-f_g(r)]\sin^2(\theta)\\
&B_{4341}=0 && B_{4231}=-\frac{\zeta(r)}{2f_g(r)}\sin(2\theta) && B_{3231}=0\\
&B_{4332}=0 && B_{4221}=0 && B_{3221}=0\\
&B_{4331}=0 && B_{4141}=\tfrac{1}{2}c^2f_g''(r) && B_{3131}=-\frac{rf_g'(r)}{2f_g(r)}\sin^2(\theta)\\
&B_{4321}=\frac{\zeta(r)}{2f_g(r)}\sin(2\theta) && B_{4132}=-\frac{\zeta(r)}{f_g(r)}\sin(2\theta) && B_{3121}=0\\
&B_{4242}=\tfrac{1}{2}c^2rf_g(r)f_g'(r) && B_{4131}=-\frac{r^2}{2\zeta(r)}\frac{d}{dr}\left[\frac{\zeta^2(r)}{r^2f_g(r)}\right]\sin^2(\theta) && B_{2121}=-\frac{rf_g'(r)}{2f_g(r)}
\end{aligned} \tag{7.11}$$

8. Examples of Particle Solutions

Example 1. A charged particle with total charge $q_{\mathrm{T}}=q$.

$$f_e(r)=q(4\pi\epsilon_0r^2)^{-1}\{1-\exp[-(r/r_0)^3]\} \tag{8.1a}$$

$$f_\epsilon(r)=1+[-1+(\lambda_m\{1+\lambda_c\}-9)(r/r_0)^3]\exp[-(r/r_0)^3] \tag{8.1b}$$

$$h(r)=\gamma\epsilon_0(qr)^{-1}\{1+[-1+(\lambda_m\{1+\lambda_h\}-9)(r/r_0)^3]\exp[-(r/r_0)^3]\} \tag{8.1c}$$

$$m_0c^2=q^2\Gamma(2/3)\lambda_m(1+\lambda_c)(3-3\cdot 2^{1/3}+3^{1/3})(54\pi\epsilon_0r_0)^{-1}\approx 5.287980438\cdot 10^{-3}q^2\lambda_m(1+\lambda_c)(\epsilon_0r_0)^{-1} \tag{8.1d}$$

$$q_{\mathrm{T}}=q \tag{8.1e}$$

$$\boldsymbol{J}_{\mathrm{T}}=\gamma m_0(1+\lambda_h)[2q(1+\lambda_c)]^{-1}\boldsymbol{e}_z \tag{8.1f}$$

$$\boldsymbol{M}_{\mathrm{T}}=\tfrac{2}{3}\gamma\boldsymbol{e}_z \tag{8.1g}$$

If we set the z-component of $\boldsymbol{J}_{\mathrm{T}}$ to $\frac{1}{2}\hbar$ for spin $\frac{1}{2}$ particles, then from (5.10b) and (8.1f) we obtain a generalized magneton result

$$2\mu_{\mathrm{m}} = \gamma = \frac{q\hbar(1+\lambda_c)}{m_0(1+\lambda_h)} \tag{8.2}$$

where the the quantum spin factor $g_s = 2$ appears automatically.

From (8.1b) we expect a structural transition in the vicinity of $\lambda_m(1+\lambda_c) = 9$. The effects can be seen in the graphs in §9 where it seems reasonable to identify $\lambda_m(1+\lambda_c) \ll 9$ with leptons and $\lambda_m(1+\lambda_c) \gg 9$ with baryons. Note that the mass m_0 (8.1d) depends on the ratio $\lambda_m(1+\lambda_c)/r_0$. Thus there could be high mass leptons and low mass baryons depending on the value of r_0. Obviously $\lambda_m(1+\lambda_c)$ can be chosen such that the rest mass is negative or zero. Even if we have a positive rest mass, there can be regions in the interior of a particle in which the energy density $En(r)$ is negative, as can be seen in the graphs. Within the context of this simple model, that seems to be unavoidable for leptons.

Example 2. A particle that has mass and intrinsic angular momentum; but the total charge and total magnetic moment are zero, thus giving $\boldsymbol{F}_{\mathrm{T}} = 0$ and $\boldsymbol{W}_{\mathrm{T}} = 0$. Therefore, the lack of an interaction with an external magnetic field does not rule out the existence of intrinsic angular momentum. In the lepton region $\lambda_m(1+\lambda_c) \ll \frac{19}{8}$, we expect this to be a model for neutrinos. In the baryon region $\lambda_m(1+\lambda_c) \gg \frac{19}{8}$, there are two possibilities. One is a stable particle with spin $\frac{1}{2}$ that is as elusive as the neutrino. The other is an uncharged particle with zero spin if $\gamma = 0$ or $\lambda_h = -1$.

$$f_e(r) = q(4\pi\epsilon_0 r_0^2)^{-1}(r/r_0)^3 \exp[-(r/r_0)^3] \tag{8.3a}$$

$$f_\epsilon(r) = 1 + [-1 + \{\lambda_m(1+\lambda_c\} - \tfrac{19}{8})(r/r_0)^3] \exp[-(r/r_0)^3] \tag{8.3b}$$

$$h(r) = \gamma\epsilon_0(qr)^{-1}\{1 + [-1 + (\lambda_m\{1+\lambda_h\} - \tfrac{19}{8})(r/r_0)^3] \exp[-(r/r_0)^3]\} \tag{8.3c}$$

$$m_0c^2 = q^2\lambda_m(1+\lambda_c)(243\pi\epsilon_0 r_0)^{-1} \approx 1.309917227 \cdot 10^{-3} q^2\lambda_m(1+\lambda_c)(\epsilon_0 r_0)^{-1} \tag{8.3d}$$

$$q_{\mathrm{T}} = 0 \tag{8.3e}$$

$$\boldsymbol{J}_{\mathrm{T}} = \gamma m_0(1+\lambda_h)[2q(1+\lambda_c)]^{-1}\boldsymbol{e}_z \tag{8.3f}$$

$$\boldsymbol{M}_{\mathrm{T}} = 0 \tag{8.3g}$$

9. Graphs and Further Discussion of Particle Solutions

We will define the normalized radius r_N and the following normalized functions where $\rho(r)$ is calculated from (5.2b) and $En(r)$ from (5.2g).

$$r_N = r/r_0 \qquad f_{eN}(r_N) = \epsilon_0 q^{-1} r_0^2 f_e(r_N r_0)$$
$$f_{\epsilon N}(r_N) = f_\epsilon(r_N r_0) \qquad \rho_N(r_N) = q^{-1} r_0^3 \rho(r_N r_0) \qquad En_N(r_N) = \epsilon_0 q^{-2} r_0^4 En(r_N r_0) \tag{9.1}$$

Figure 1 is a plot of the normalized electric field f_{eN} vs. normalized radius r_N for a charged particle. If we take the 2014 CODATA [7] value for the proton rms charge radius as an approximation for r_0, then (8.1d) gives $\lambda_m(1+\lambda_c) = 8404$. Figures 2 - 4 show the resulting plots for $f_{\epsilon N}$, ρ_N and En_N vs. normalized radius r_N. In Figure 3, there is a zero at $r_N = 1.13075$. The total charge q is the sum of the charges in each region $(2186.18, -2185.18)q$. Note that the charge density in the inner region has the same sign as q.

There is no accepted value for the radius of an electron. It seems to act like a point particle. This is mirrored theoretically by the fact that for $\lambda_m(1+\lambda_c) < 0.0001$, the results for $\rho(r)$ are within 0.002% of the results for $\lambda_m(1+\lambda_c) = 0$. For an electron, this corresponds to $r_0 \approx 2 \cdot 10^{-20}$m. Figures 5 - 7 show the results. In Figure 6, there is a zero at $r_N = 1.08374$. The total charge q is the sum of the charges in each region $(-1.79127, 2.79127)q$. Note that the charge density in the inner region has the opposite sign as q. In Figure 7, the total mass is zero since $\lambda_m(1+\lambda_c) = 0$, even though visually that does not appear to be true. Setting $\lambda_m(1+\lambda_c) = 0.0001$ does not change the visual appearance of the Figure 7, though it does give a non-zero value for the total mass which is about 0.004% of the magnitude of the mass in each region.

Comparing Figures 2 - 4 with Figures 5 - 7, we see the effects of the structural transition mentioned in §8. In particular, the structure of regions of charge with alternating signs seems to be the classical field theory equivalent to the quantum field theory concept of a bare charge screened by vacuum polarization. However the behavior below and above the transition is different. Below the transition, the inner charge is surrounded by an outer layer of charge which is greater in magnitude than the magnitude of the inner charge. Above the transition, the inner charge is surrounded by an outer layer of charge which is less in magnitude than the magnitude of the inner charge.

Since the charge of an electron is negative and the charge of a proton is positive, this model predicts that there is a central core of positive charge and an outer region of negative charge for both the electron and the proton. Since the electron is below the structural transition and the proton is above it, perhaps the structural transition separates the leptons from the baryons.

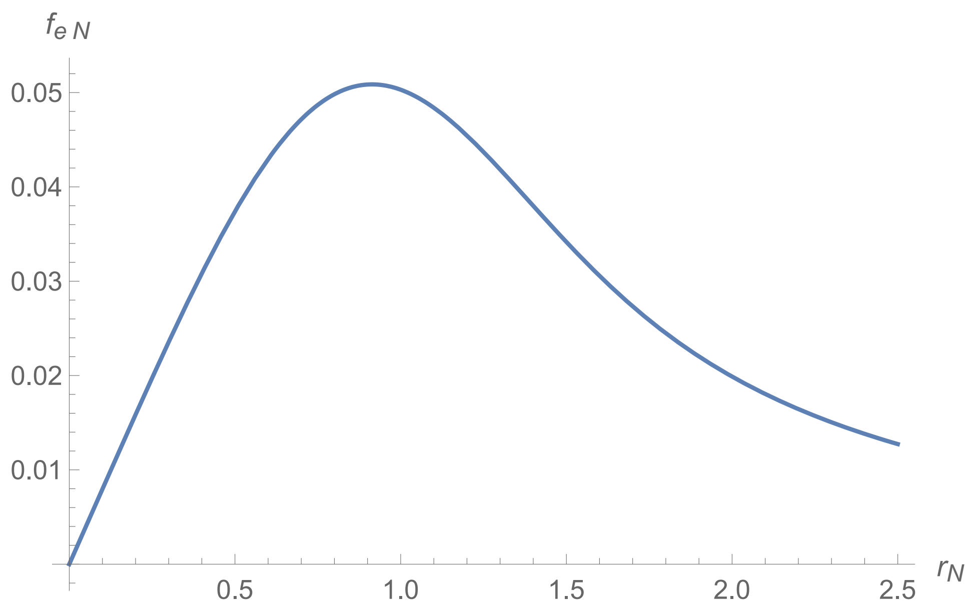

Figure 1: Normalized Electric Field vs. Normalized Radius.

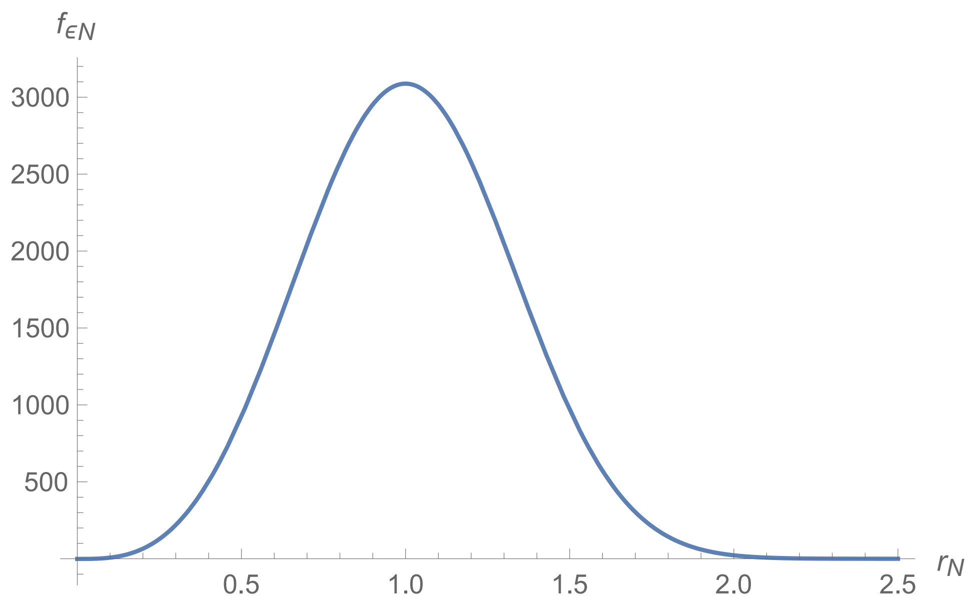

Figure 2: Function f_ϵ vs. Normalized Radius, $\lambda_m(1+\lambda_c) = 8404$.

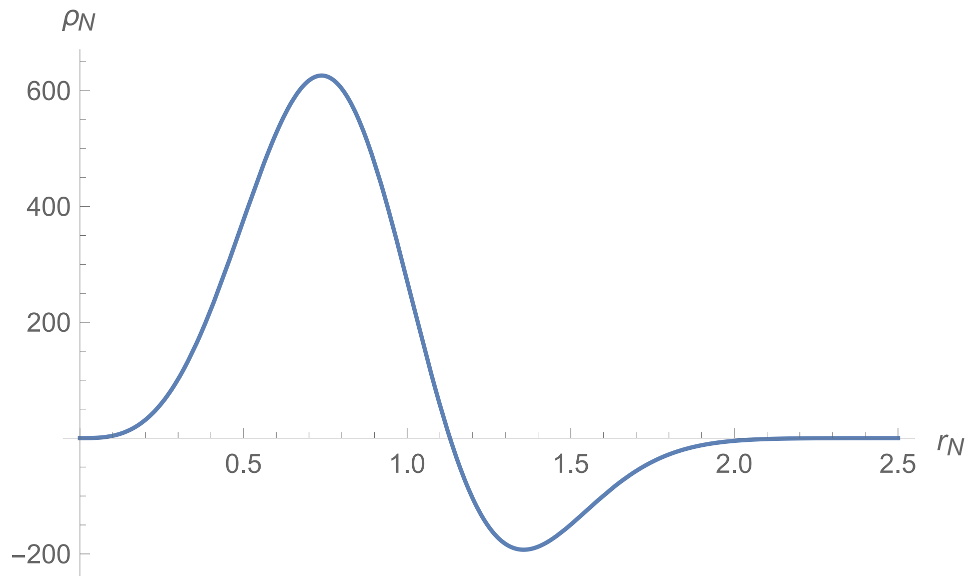

Figure 3: Normalized Charge Density vs. Normalized Radius, $\lambda_m(1+\lambda_c) = 8404$.
$q = (2186.18 - 2185.18)q$

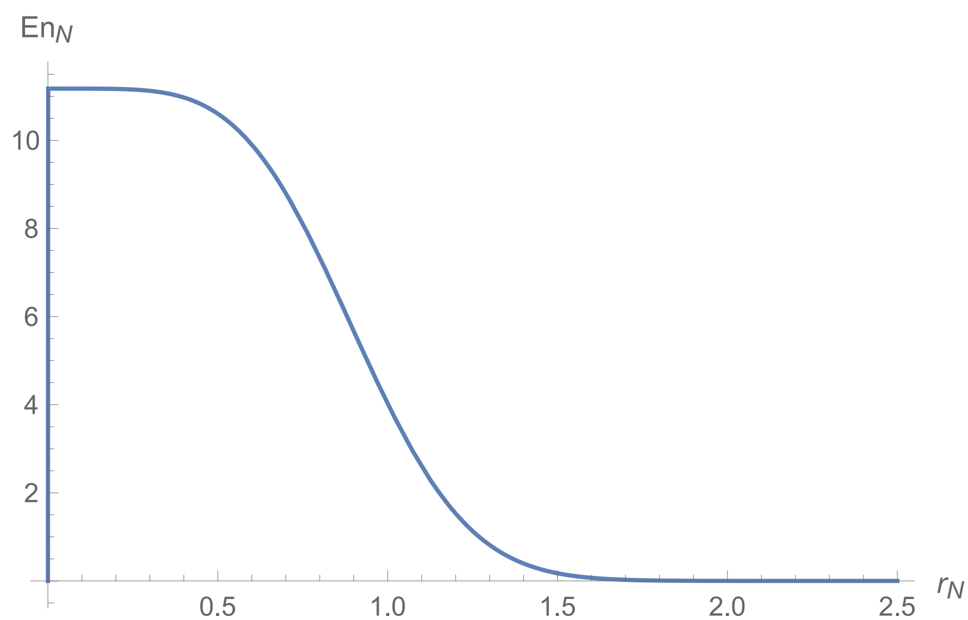

Figure 4: Normalized Energy Density vs. Normalized Radius, $\lambda_m(1+\lambda_c) = 8404$.

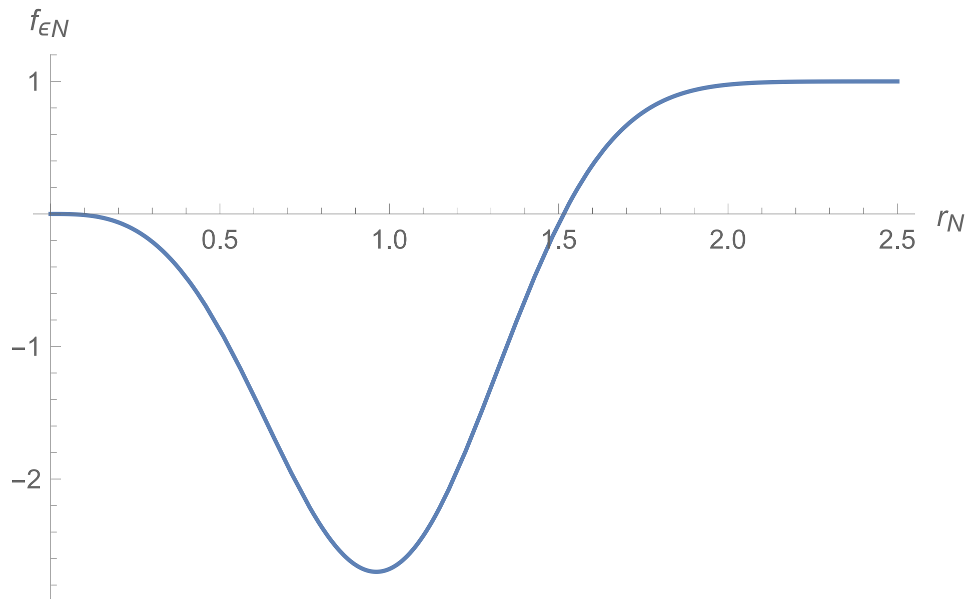

Figure 5: Function f_ϵ vs. Normalized Radius, $\lambda_m = 0$.

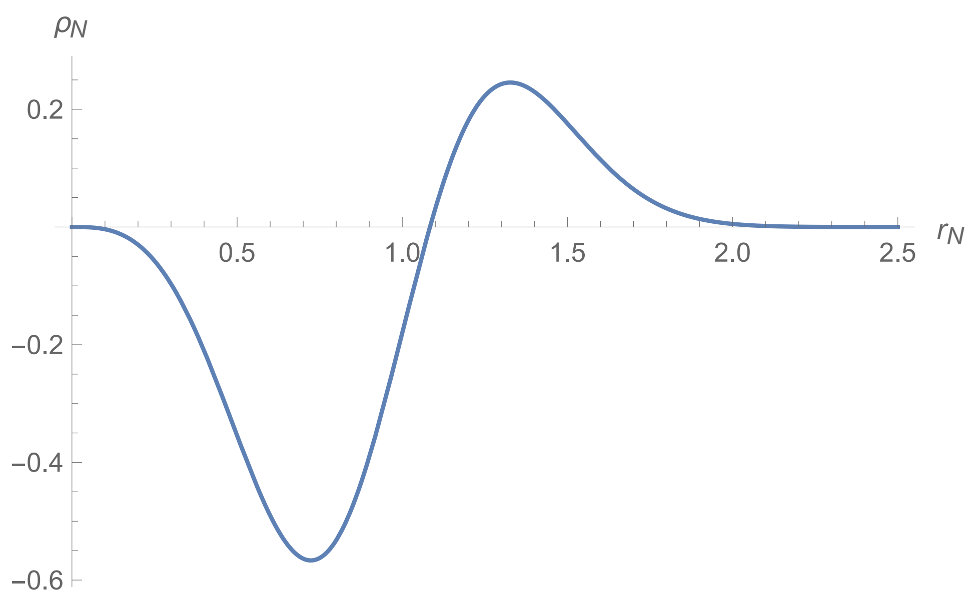

Figure 6: Normalized Charge Density vs. Normalized Radius, $\lambda_m = 0$.
$q = (-1.79127 + 2.79127)q$

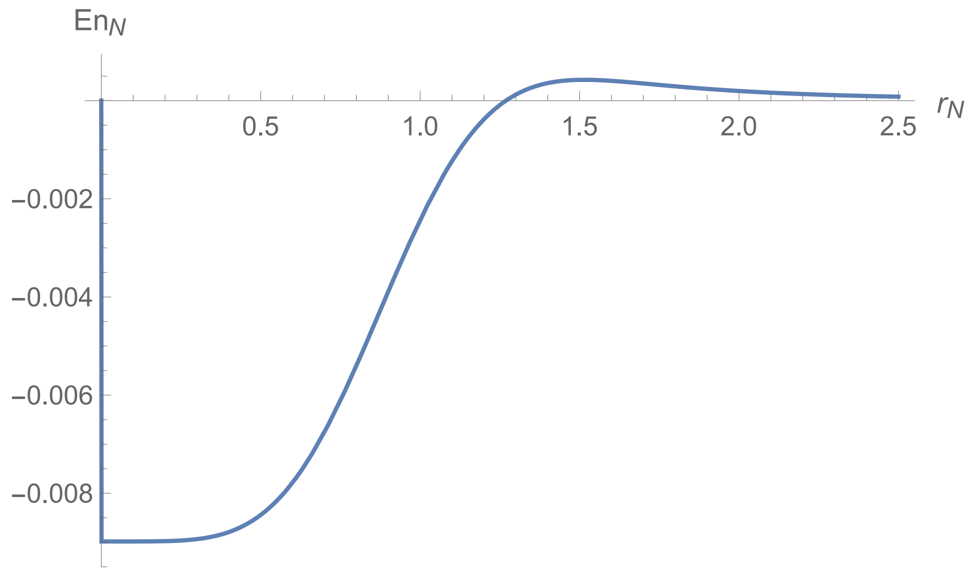

Figure 7: Normalized Energy Density vs. Normalized Radius, $\lambda_m = 0$.
The total mass m_0 is zero since $\lambda_m = 0$ (8.1d).

10. Electromagnetic and Gravitational Waves

In a curved space with a Peres [8, 9] type of cylindrically symmetric metric

$$ds^2 = dr^2 + r^2 d\varphi^2 + dz^2 - c^2 dt^2 - [f'(z-ct)]^2 f_g(r)(dz - cdt)^2 \tag{10.1}$$

there exist electromagnetic waves and gravitational waves that are independent of each other and that couple to different terms in the non-Riemannian part of the curvature tensor. Let

$$A_\mu = cf(z-ct)f_{em}(r)(1,0,0,0) \tag{10.2a}$$
$$\chi_{\mu\nu\rho\sigma} = \epsilon_0(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}) \tag{10.2b}$$
$$Q = 0 \tag{10.2c}$$
$$c^2 a^1_{33} = -ca^1_{43} = a^1_{44} = [f'(z-ct)]^2 f_a(r) \tag{10.2d}$$
$$a^1_{42} = -ca^1_{32} = -(c/2)f'(z-ct)f_b(r) \tag{10.2e}$$
$$a^2_{41} = -ca^2_{31} = c(2r)^{-1}f'(z-ct)f'_b(r) \tag{10.2f}$$

where $a^\mu_{\nu\sigma}$ obeys the constraints (4.9). Then the non-zero components of $T_{\mu\nu}$, $G_{\mu\nu}$ and $S_{B\mu\nu}$ are

$$T_{44} = -cT_{34} = c^2 T_{33} = c^4\epsilon_0[f'(z-ct)]^2 f^2_{em}(r) \tag{10.3a}$$
$$G_{44} = -cG_{34} = c^2 G_{33} = c^2(2r)^{-1}[f'(z-ct)]^2[f'_g(r) + rf''_g(r)] \tag{10.3b}$$
$$S_{B44} = -cS_{B34} = c^2 S_{B33} = [f'(z-ct)]^2[f'_a(r) + r^{-1}f_a(r) + c^2(2r)^{-1}f_b(r)f'_b(r)] \tag{10.3c}$$

Thus $T^{\mu\nu}{}_{;\nu} = 0$ and $S_B^{\mu\nu}{}_{;\nu} = 0$ and equations (4.11) reduce to

$$f_a(r) = 8\pi G\epsilon_0 r^{-1}\int_0^r dr' r' f^2_{em}(r') \tag{10.4a}$$
$$f_g(r) = \tfrac{1}{2}\int_r^\infty dr'(r')^{-1} f^2_b(r') \tag{10.4b}$$

Note that the electromagnetic wave does not depend on $f_g(r)$. If $f_g(r) = 0$, the metric reduces to the flat space cylindrical metric and there is no gravitational wave. If$f_{em}(r) = 0$, there is a gravitational wave but no electromagnetic wave.

For this type of electromagnetic wave, $j_\mu \neq 0$, but $j_\mu j^\mu = 0$ and

$$\begin{aligned} j^\mu_{\mathrm{T}} &= \int_0^\infty dr\, r \int_0^{2\pi} d\varphi \int_{-\infty}^\infty dz\, j^\mu \\ &= 2\pi\epsilon_0 \left.[r f_{em}(r)]\right|_{r=0}^\infty \left. f(z-ct)\right|_{z=-\infty}^\infty (0,\ 0,\ c,\ 1) \\ &= 0 \end{aligned} \tag{10.5}$$

In free space, electromagnetic waves are usually assumed to have zero current, $j_\mu = 0$. However, if we admit the possibility of non-zero field currents such that $j_\mu j^\mu = 0$, then we have a class of force-free wave solutions that have a spatial variation in the plane perpendicular to the direction of propagation. These null-vector field currents are a generalization of the displacement currents in standard circuit theory. They are intrinsic to the structure of the wave; they are not an external source. Furthermore, this class of electromagnetic solutions introduces terms only in the non-Riemannian part of the curvature tensor. In that sense, the energy in these solutions does not add to the total gravitational mass of the universe. The energy in the gravitational waves does affect the Riemannian curvature.

As examples, consider

$$f_{em}(r) = r\exp(-\beta^2 r^2) \tag{10.6a}$$
$$f_a(r) = \pi G\epsilon_0 (r\beta^4)^{-1}[1-(1+2\beta^2r^2)\exp(-2\beta^2r^2)] \tag{10.6b}$$
$$f_b(r) = r\exp(-\beta^2 r^2) \tag{10.6c}$$
$$f_g(r) = (8\beta^2)^{-1}\exp(-2\beta^2r^2) \tag{10.6d}$$

The physical question is whether these examples describe the local structure of classical waves or whether they describe waves that are different than classical waves. If they are different, then the question is how can they be generated and detected.

For waves, the components of the curvature tensor $B_{\mu\nu\rho\sigma}$ are

$$\begin{array}{lll} B_{4343} = 0 & B_{4241} = \tfrac{1}{2}c^2 r f_b'(r) f''(\eta) & B_{4121} = \tfrac{1}{2}c r^{-1}[f_b(r) - r f_b'(r)]f'(\eta) \\ B_{4342} = \tfrac{1}{4}c^2 f_b(r) f_g'(r)[f'(\eta)]^3 & B_{4232} = -c^{-1}BI & B_{3232} = c^{-2}BI \\ B_{4341} = 0 & B_{4231} = \tfrac{1}{2}c f_b(r) f''(\eta) & B_{3231} = \tfrac{1}{2} r f_b'(r) f''(\eta) \\ B_{4332} = 0 & B_{4221} = 0 & B_{3221} = 0 \\ B_{4331} = 0 & B_{4141} = BII & B_{3131} = c^{-2}BII \\ B_{4321} = 0 & B_{4132} = \tfrac{1}{2}c f_b(r) f''(\eta) & B_{3121} = -\tfrac{1}{2} r^{-1}[f_b(r) - r f_b'(r)]f'(\eta) \\ B_{4242} = BI & B_{4131} = -c^{-1}BII & B_{2121} = 0 \end{array} \tag{10.7}$$

where

$$\begin{aligned} \eta &= z - ct \\ BI &= \tfrac{1}{4} r\{4 f_a(r) + c^2[f_b(r) f_b'(r) + 2 f_g'(r)]\}[f'(\eta)]^2 \\ BII &= \tfrac{1}{4} r^{-1}\{4 r f_a'(r) + c^2[f_b(r) f_b'(r) + 2 r f_g''(r)]\}[f'(\eta)]^2 \end{aligned} \tag{10.8}$$

The equations for the paths are

$$0 = \frac{d^2r}{d\xi^2} - r\left(\frac{d\varphi}{d\xi}\right)^2 + f_b(r)f'(\eta)\frac{d\varphi}{d\xi}\left(\frac{dz}{d\xi} - c\frac{dt}{d\xi}\right) + \left[\frac{f_a(r)}{c^2} + \frac{f_g'(r)}{2}\right][f'(\eta)]^2\left(\frac{dz}{d\xi} - c\frac{dt}{d\xi}\right)^2 \tag{10.9a}$$

$$0 = \frac{d^2\varphi}{d\xi^2} + \frac{2}{r}\frac{dr}{d\xi}\frac{d\varphi}{d\xi} - \frac{f_b'(r)f'(\eta)}{r}\frac{dr}{d\xi}\left(\frac{dz}{d\xi} - c\frac{dt}{d\xi}\right) \tag{10.9b}$$

$$0 = \frac{d^2z}{d\xi^2} - f_g'(r)[f'(\eta)]^2\frac{dr}{d\xi}\left(\frac{dz}{d\xi} - c\frac{dt}{d\xi}\right) - f_g(r)f''(\eta)\left(\frac{dz}{d\xi} - c\frac{dt}{d\xi}\right)^2 \tag{10.9c}$$

$$0 = \frac{d^2t}{d\xi^2} - \frac{f_g'(r)[f'(\eta)]^2}{c}\frac{dr}{d\xi}\left(\frac{dz}{d\xi} - c\frac{dt}{d\xi}\right) - \frac{f_g(r)f''(\eta)}{c}\left(\frac{dz}{d\xi} - c\frac{dt}{d\xi}\right)^2 \tag{10.9d}$$

11. Conclusions

We have modified the Einstein-Maxwell equations by adding three types of terms and have constructed various particle and wave solutions. The solutions are force-free and mathematically well-behaved. The details of the construction are arbitrary so long as they obey the boundary conditions. The particle solutions have some of the properties required for the elementary particles. We have also shown that the curvature terms arising from the non-metric components of a general symmetric connection couple in various ways to the particle solutions and to the electromagnetic and gravitational wave solutions.

Acknowledgments

Many of the calculations were done using *Mathematica*® [10] with the *MathTensor*™ Application Package [11].

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