Navier Stockes equation, integrals of motion and Generalization of the equation of continuity of the flow of matter to the Theory of Relativity

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Abstract

The use of N-S equation is of outmost important for everyday life: airplanes, ships, underwater ships, etc. So, the Clay Institute promises 1 000 000 dollars for a good solution. Present paper is about Estonian author confidence, that he have solved the problem.

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Areas of physics: mathematical physics, the science of matter, field theory. A new formula for the continuity of the flow of matter is derived. The existing theory suffers from the "problem of uniqueness": it is not always clear what states of matter are real.

Intuitively, I think that if you throw a handful of balls from the bearing, the number of balls is invariably preserved. No matter how strong gravity would be observed. Therefore, in addition to the Einstein equations, an additional law must be fulfilled that keeps the number of balls unchanged. Looking ahead, I will say that it has a simple appearance:

$$\sum_{\nu=0}^{3} J^{\nu}_{;\nu} = 0\,, \qquad J^{\nu} = \rho\, u^{\nu}\,. \tag{3f}$$

Here the density of matter is multiplied by its four-dimensional velocity, and the resulting flux has a zero covariant divergence

$$\sum_{\nu=0}^{3} J^{\nu}_{;\nu} = \sum_{\mu=0}^{3} \frac{\partial J^{\mu}}{\partial x^{\mu}} + \sum_{\alpha=0}^{3} \sum_{\nu=0}^{3} \Gamma^{\nu}_{\nu\,\alpha}\, J^{\alpha}\,,$$

where $\Gamma^{\nu}_{\mu\,\alpha}$ denotes the "connectivity coefficients", also known as the "Christoffel symbols". They are calculated according to the "metric" of space-time in a known way. In the case of Minkowski space-time, the metric has the form of the diagonal matrix $g_{\nu\mu} = \mathrm{diag}(-1,\, 1,\, 1,\, 1)$, so that the square of the linear element is $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. This is the interval between two very close points. Recall that by the Pythagorean theorem $dL^2 = dx^2 + dy^2 + dz^2$.

Therefore, in the case of choosing a flat Minkowski space-time (or, alternatively: [2]), there is a simple and well-known formula for the continuity of the flow of matter:

$$\sum_{\nu=0}^{3} J^{\nu}_{,\nu} = \sum_{\nu=0}^{3} \frac{\partial J^{\nu}}{\partial x^{\nu}} = 0\,, \tag{1f}$$

it is well known that in the case of such a space-time, all $\Gamma^{\nu}_{\mu\,\alpha} = 0$. However, here I give a generalization to high velocities of "balls" (for example, protons in a particle accelerator), and not only the well-known classical theory that gives

$$\frac{\partial \rho}{\partial t} + \nabla\,(\rho\, \vec{u}) = 0\,.$$

So, our innovation is the preservation of not only the number of balls, but also the total energy of the system. This innovation is a relativistic generalization of the law of continuity to the Special Theory of Relativity. Let us now generalize it to the General Theory of Relativity.

It can be shown from the energy-momentum tensor of the "ideal fluid" [1] that there is

$$\sum_{\nu=0}^{3} J^{\nu}_{;\nu} = -p \sum_{\nu=0}^{3} u^{\nu}_{;\nu} \,, \tag{2f}$$

where the fluid pressure is multiplied by the divergence from the fluid velocity. Recall that by choosing Minkowski space, I have zero Christoffel symbols and therefore

$$\sum_{\nu=0}^{3} J^{\nu}_{,\nu} = -p \sum_{\nu=0}^{3} u^{\nu}_{,\nu} \,.$$

However, in the Minkowski space the formula (1f) is known. Therefore, I must assume that the mathematically verified state of an ideal fluid has zero pressure, $p = 0$, and then from (2f)

$$\sum_{\nu=0}^{3} J^{\nu}_{;\nu} = 0 \,.$$

So, our assumption was confirmed.

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[1] Lightman AP, Press WH, Price RH, Teukolsky SA. Problem Book in Relativity and Gravitation. Princeton, Princeton University Press, 1975.

[2] By transforming the coordinates at any chosen point, you can reset all the Christoffel symbols (see "Theory of the Field" by Landau-Lifshitz), so at this point the formula will give a restriction on the state of matter.

I. "PERFECT FLUID" MODEL IS NOT MATHEMATICALLY CONSISTENT

Derived the integral of motion for perfect fluid. It reduces the number of valid equations of state: must be $p = 0$. This can be regarded as part of the solution of more general problem: the fluid with viscosity, which is also showing $p = 0$ by another approach.

The energy-momentum tensor of the perfect fluid is

$$T^{\nu\mu} = (\rho + p)\, u^{\nu}\, u^{\mu} + p\, g^{\nu\mu} \,. \tag{1}$$

with $u^{\nu}\, u_{\nu} = -1$. Then the $T^{\nu\mu}_{;\nu} = 0$ means

$$0 = u_{\mu}\, T^{\nu\mu}_{;\nu} = \frac{d\rho}{d\tau} + (\rho + p)\Theta \,, \tag{2}$$

where

$$\Theta = u^{\nu}_{;\nu}\,, \qquad \frac{d\rho}{d\tau} = \frac{\partial\rho}{\partial x^{\nu}}\,u^{\nu}\,. \tag{3}$$

Let us denote

$$J^{\mu} = -T^{\nu\mu}\,u_{\nu} = \rho\,u^{\mu}\,. \tag{4}$$

Then

$$J^{\mu}_{;\mu} = \frac{d\rho}{d\tau} + \rho\,\Theta\,, \tag{5}$$

and so from Eq.(16)

$$J^{\mu}_{;\mu} = -p\,\Theta\,. \tag{6}$$

While solving the problems in Special Relativity one holds the background spacetime fixed: Minkowskian, no need of General Relativity Equation $G^{\nu\mu} = 8\pi\,T^{\nu\mu}$ then. Such method, applied to Dark Matter, can solve even it: [4]. If in a model the spacetime is flat and fixed Minkowskian, then holds exactly

$$J^{\mu}_{,\mu} = -p\,\Theta\,, \tag{7}$$

where $\Theta = u^{\nu}_{,\nu}$.

But is known, that in flat spacetime $J^{\mu}_{,\mu} = 0$. Therefore holds 1) $p = 0$ or 2) $\Theta = 0$ with $\rho = \text{const}$. And so in addition to the known $T^{\nu\mu}_{;\nu} = 0$, by fact holds for the perfect fluid following formula:

$$J^{\mu}_{;\mu} = 0\,. \tag{8}$$

One can demonstrate (viXra:1711.0272, viXra:1304.0086), that in latter case holds

$$\int J^{t}\,\sqrt{-g}\,dV = \text{const}\,,$$

where $J^{t} = \rho\,u^{t}$. Therefore, the conserved is not the rest-mass ρ, but the energy $\rho\,u^{t} = \rho\,c^{2}/\sqrt{1-(v/c)^{2}}$. Therefore, the $\rho \neq \text{const}$, so the Θ can not be zero, and the only one mathematically consistent possibility remains: the pressure-free dust with $p = 0$.

II. THE ANSWER TO MILLENNIUM PRIZE PROBLEM

One might argue, that between a planet and the vacuum is discontinuity of measurements. Thus, that place violates the strong equivalence principle, to avoid it, one must understand that there are no discontinuities in Nature. The problematic places are having the thin transitional areas. So, there are regular functions $f(t,x,y,z)$, their derivations are all continuous. Then, the Taylor series at initial moment $t=0$ imply, that the Navier Stockes equations can not be the source of divergency:

$$|\sum f^{(k)}\,\frac{t^k}{k!}| < \sum |f^{(k)}|\,\frac{t^k}{k!} < M\sum\,\frac{t^k}{k!} < \infty\,, \tag{9}$$

where M is the maximum derivative at initial moment. Then, if the measurable-s (velocity, density, etc) are regular at initial moment (thus, the physical), then it is regular and smooth all the future and satisfies the N-S equations:

The NS equation has form $N(t,x,y,z) = 0$ for all t, look Eq.(13). Therefore, I have following equations at $t=0$

$$n_k := \frac{\partial^k N}{\partial t^k}\Big|_{t=0} = 0\,, \tag{10}$$

for all $k = 1,2,3,\ldots$. On the other hand, one inserts the Taylor series

$$f = \sum f^{(k)}\,\frac{t^k}{k!}\,, \tag{11}$$

where f can be density ρ, pressure p, velocity $\vec{v}$, viscosity μ etc. One inserts them all into NS equation $N(t,x,y,z) = 0$, and collects the terms with the same power of the t

$$N = N_0 + N_1\,t + N_2\,t^2 + N_3\,t^3 + \ldots\,. \tag{12}$$

It turned out, what the structure of NS equation is so lucky (obviously in contrary to [1]), what all $N_k \sim n_k = 0$, thus all $N_k = 0$.

A. The form of NS equation

Is well known, what Navier-Stokes equations are derived to have such simple form [2]

$$0 = N(t,x,y,z) := -\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\,\nabla)\vec{v}\right) + \rho\vec{F} - \nabla\,p + (\gamma+\mu)\,\nabla\,(\mathrm{div}\,\vec{v}) + \mu\,\triangle\vec{v}\,, \tag{13}$$

with equation of state $p = p(\rho,T)$, the dissipative constants γ, μ are assumed to be constant while derivation of latter case of NS equation.

Let's now the γ and μ are functions of space and time. Then the NS equation $N = 0$ has becomes [3]

$$N := -\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\,\nabla)\vec{v}\right) + \tag{14}$$

$$+\rho\vec{F} - \nabla\, p + (\gamma+\mu)\,\nabla\,(\mathrm{div}\,\vec{v}) + \mu\,\triangle\vec{v} +$$

$$+A\,\nabla v^i + B^i\,\mathrm{div}\,\vec{v} + C_k\,\nabla v^k\,,$$

where $A := \nabla\mu$, $B := \nabla\gamma$, the vector k-th component is $C_k := (\nabla\mu)_k$.

III. ON GENERAL SOLUTION

Let the viscous coefficients are time and space functions, e.g. $\eta = \eta(t,x,y,z)$. If the fluid is electrically neutral, then the potential field, which acts on the fluid is zero: $\vec{U} = 0$, nevertheless the fluid can experience the pushing from the sides of the fluid (the wings of airplane are pushing air around the plane).

The norm of 4-velocity is $u^\nu u_\nu + 1 = 0$. Then by taking the covariant gradient, one gets

$$0 = (u^\nu u_\nu + 1)_{;\alpha}\, u^\alpha = a^\nu u_\nu + u^\nu a_\nu = 2\, a^\nu u_\nu\,, \tag{15}$$

where 4-acceleration $a^\nu = u^\nu_{;\alpha}\, u^\alpha$.

The 4-current density is

$$J^\nu = -T^{\nu\mu}\, u_\mu = \rho\, u^\nu\,, \tag{16}$$

where the energy-momentum tensor $T^{\nu\mu}$ of viscous fluid is from book of Lightman. Then the

$$J^\nu_{;\nu} = d\rho/d\tau + \rho\,\Theta\,, \tag{17}$$

where $\Theta = u^\nu_{;\nu}$. If holds Eq.(25), then the relative density rate $((d\rho/d\tau)/\rho)$ is the 4-divergence Θ, that is in perfect accordance with divergence physical meaning (the counter of the field sources).

But on the other hand, because $T^{\nu\mu}_{;\nu} = 0$

$$(-T^{\nu\mu}\, u_\mu)_{;\nu} = -T^{\nu\mu}\, u_{\mu\,;\nu} = -\beta + \eta\, a^\nu a_\nu\,, \tag{18}$$

where

$$\beta = p\,\Theta + (2\eta/3 - \zeta)\,\Theta^2 - 2\,\eta\, u_{\nu\,;\mu}\, u^{(\nu\,;\mu)}\,, \tag{19}$$

where $2\, u^{(\nu\,;\mu)} = u^{\nu\,;\mu} + u^{\mu\,;\nu}$.

$$u_\mu\, T^{\nu\mu}_{;\nu} = -d\rho/d\tau - \rho\,\Theta - \beta = 0 \tag{20}$$

While the derivations the following facts were used:

$$0 = (u^\beta u_{\beta\,;\alpha})^{;\alpha} = u^{\beta\,;\alpha} u_{\beta\,;\alpha} + u^\beta u^{;\alpha}_{\beta\,;\alpha}\,, \tag{21}$$

$$a^{;\alpha}_{\alpha} = (u^\beta u_{\alpha\,;\beta})^{;\alpha} = u^{\beta\,;\alpha} u_{\alpha\,;\beta} + u^\beta u^{\alpha}_{;\beta\,;\alpha}\,. \tag{22}$$

Thus, from Eqs.(16)–(20) holds $a^\nu a_\nu = 0$. From the Special relativity (the Dr. Teet Örd's lectures) is known, that $a^\nu a_\nu$ is zero only if the 3-acceleration is zero: $a = (0, 0, 0)$. Latter imply, that motion is force-free, the lines of fluid are geodetic $a^\nu = 0$ at every point of spacetime. So, without experiencing any acceleration, even the acceleration of circular orbit, then the fluid is totally static and experiences no non-compensated pushing from the edges (no flying airplane then). In conclusion, the general solution (which is consistent with mathematics) of N-S equation is the pressure-free dust, $p = 0$.

A. Case of zero viscosity

It has $\eta = \zeta = 0$. Then from Eqs.(16)–(20)

$$d\rho/d\tau + (\rho + p)\,\Theta = 0\,. \tag{23}$$

Then from Eq.(17) I have

$$J^\nu_{;\nu} = -p\,\Theta\,, \tag{24}$$

Because in weak gravity limit $J^\nu_{;\nu} \to J^\nu_{,\nu} = 0$, but $\Theta \to u^\nu_{,\nu}$ does not turn to zero, then must be $p = 0$.

IV. ON THE COVARIANT DIVERGENCE OF CURRENT DENSITY

As you have seen, the mathematically consistent solution in case of zero viscosity must have

$$J^\nu_{;\nu} = 0\,, \tag{25}$$

But the N-S equations does not satisfy it. Then let us agree, that latter condition is necessary also for viscous fluid.There is Gauss theorem in curved spacetime (viXra:1711.0272, viXra:1304.0086), latter produces formula, which is easy to demonstrate by a math-software. In case $A^\nu_{;\nu} = 0$ and isolated field $A^\nu = 0$, $r > r_0$ it simplifies

$$\int A^t \sqrt{-g}\, dxdydz = \text{const}\,, \tag{26}$$

Applying to current density in non-relativistic case this constant is the conservation of the fluid mass-energy: $\int \rho \, dxdydz = \text{const}$. This law of (energy) conservation is very important result, because there is the problem of Energy concept in General Relativity: viXra:1306.0012.

Where $\sqrt{-g} = 1$ for Minkowski as fixed background spacetime, notably the variation principle can be used following way: to fix spacetime and let matter assume the optimal energy level: [4], vixra.org/abs/1512.0347. Let us study one obvious solution of above integral equation, in educational purposes

$$A^t \sqrt{-g} = b(x, y, z)\,. \tag{27}$$

Thus, in addition to above formulas holds

$$J^t = \rho \, u^t = b(x, y, z)\,. \tag{28}$$

Because $u^t = c/\sqrt{1-(v/c)^2} \approx c$, then in non-astrophysical situations one can write $\rho = b(x, y, z)$. Remember, that the ρ measures the co-moving observer, so, there can be $d\rho/d\tau \neq 0$. The non-changing ρ means the non-changing u^ν, so the general solution of N-S is stationary. However the $a^\nu = D\,u^\nu/d\tau$ can be non-zero. Please note, that latter consideration holds also for the collapse of dust cloud, and to the dynamics of Universe (viXra:1304.0086).

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