The Bilateral Laplace Transform of the Positive Even Functions and a Proof of Riemann Hypothesis

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Abstract

We show that some interesting properties of the bilateral Laplace transform of even and positive functions both on the line $z_0 = x + iy_0$ and on a circle. We also show the Riemann hypothesis is true using these properties. We do not prove well-known theorems and encourage readers to refer to the literatures.

1. Introduction

We begin with the definition of the bilateral Laplace transform.[1] The definition of Laplace transform of a real function $\mathrm{f(t)}$ is as follows:

$$\mathrm{F(z)} = \int_{-\infty}^{\infty} f(t) \cdot e^{-zt} dt \tag{1}$$

and the inverse transform:

$$\mathrm{f(t)} = \frac{1}{i2\pi} \int_{x-i\infty}^{x+i\infty} F(z) \cdot e^{zt} dz \tag{2}$$

where $z = x + iy$.

If $\mathrm{f(t)}$ is even, then

$$\mathrm{F(z)} = \int_{-\infty}^{\infty} f(t) \cdot e^{-zt} dt = \int_{-\infty}^{\infty} f(t) \cdot e^{zt} dt = \int_{-\infty}^{\infty} f(t) \cdot \cosh(zt) dt \tag{3}$$

and since $\mathrm{F(-z) = F(z)}$, $\mathrm{F}(iy)$ is real-valued for all y. If $\mathrm{f(t)}$ is even and also positive for all t, its Laplace transform $\mathrm{F(z)}$ is transcendental.

2. Power Series expansion

It is not always easy to find the Laplace transform of a function in the closed form, but we can get the power series using the definition of the Laplace transform as follows:

$$\mathrm{F(z)} = \sum_{n=0}^{\infty} a_n \cdot z^n = a_0 + a_1 z + a_2 z^2 + \cdots \tag{4}$$

To find the coefficients of $\mathrm{F(z)}$, we examine the derivatives of (1).

[1] Since we are only dealing with the bilateral Laplace transform, The term “bilateral” will be omitted.

$$a_n = \frac{1}{n!}\int_{-\infty}^{\infty} t^n f(t)dt \tag{5}$$

Moreover, if the function $f(t) > 0$ for all t and even, all the odd terms are vanished and the power series will be like that:

$$F(z) = \sum_{n=0}^{\infty} a_{2n} \cdot z^{2n} = a_0 + a_2 z^2 + a_4 z^4 + \cdots \tag{6}$$

where $a_{2n} = \frac{1}{(2n)!}\int_{-\infty}^{\infty} t^{2n} f(t)dt$ and $a_{2n} > 0$ for all n.

Since $f(t)$ is even and positive, $F(z)$ is also even and $F(-\bar{z}) = F(\bar{z}) = \bar{F}(z)$. Hence we have $|F(z)| = |F(-z)| = |F(\bar{z})| = |F(-\bar{z})|$.

3. The convex functions

For a real function $f(x)$ and $x_1, x_2 \in \mathcal{R}$ and $\lambda \in [0,1]$, then $f(x)$ is convex if and only if

$$f[\lambda x_1 + (1-\lambda)x_2] \le \lambda f(x_1) + (1-\lambda)f(x_2).$$

and similarly, $f(x)$ is strictly convex if and only if

$$f[\lambda x_1 + (1-\lambda)x_2] < \lambda f(x_1) + (1-\lambda)f(x_2).$$

The function $f(x)$ is a midpoint convex if

$$f\left(\frac{x_1 + x_2}{2}\right) \le \frac{f(x_1) + f(x_2)}{2}$$

Further, a continuous function $f(x)$ is multiplicatively convex if and only if

$$f\left(\sqrt{x_1 x_2}\right) \le \sqrt{f(x_1) f(x_2)}$$

A multiplicatively convex function is which is increasing convex. This inequality can be obtained from the definition of the midpoint convex.

Hardy-Littlewood Theorem

Every polynomial $f(x) = \sum_{k=0}^{n} c_k x^k$ with non-negative coefficients is multiplicatively convex on $(0,\infty)$. Moreover $f(x) = \sum_{k=0}^{\infty} c_k x^k$ for $c_k \ge 0$ is strictly multiplicatively convex which is also increasing and strictly convex.

Let $F(z)$ be the Laplace transform of an even and positive function $f(t)$, then by Hardy-Littlewood, $F(z)$ is increasing and strictly convex on the x-axis of the interval $(0,\infty)$ if $F(z)$ is entire. If $F(z)$ is not entire, $F(x)$ is increasing and strictly convex in the radius of convergence. Since $F(-x) = F(x)$, $F(x)$ is symmetric at $x = 0$, and therefore $F(x)$ has a unique minimum at $x = 0$.

4. The co-positive definiteness

F(z) is complex-valued co-semipositive definite if

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n\overline{c_k}f(z_n+\overline{z_k}) \geq 0 \tag{7}$$

and similarly, $f(z)$ is complex-valued co-positive definite if

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n\overline{c_k}f(z_n+\overline{z_k}) > 0 \tag{8}$$

for all $c_n \in \mathbb{C}$ and $z_n = x_n + iy_n$.

Clearly, if a complex function $f(z)$ is complex-valued co-(semi)positive definite, then the real-valued function $f(x)$ is also co-(semi)positive definite.

Lemma 1

A real function $f(t) > 0$ for all t, then its Laplace transform is complex-valued co-positive definite.

Proof

From the definition $\mathrm{F(z)} = \int_{-\infty}^{\infty} f(t) \cdot e^{-zt}dt$

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n\overline{c_k}F(z_n+\overline{z_k}) = \sum_{n=1}^{N}\sum_{k=1}^{N} c_n\overline{c_k}\int_{-\infty}^{\infty} f(t)\cdot e^{-(z_n+\overline{z_k})t}dt$$
$$= \int_{-\infty}^{\infty} f(t)\cdot \sum_{n=1}^{N}\sum_{k=1}^{N} c_n\overline{c_k}\, e^{-(z_n+\overline{z_k})t}dt = \int_{-\infty}^{\infty} f(t)\cdot \sum_{n=1}^{N} c_n\, e^{-z_n t}\sum_{k=1}^{N} c_k\, e^{-\overline{z_k}t}dt$$
$$= \int_{-\infty}^{\infty} f(t)\cdot \left|\sum_{n=1}^{N} c_n\, e^{-z_n t}\right|^2 dt > 0$$

Lemma 2

If $f_1(z)$ and $f_2(z)$ are co-positive definite then they are also co-positive definite:

1. $\overline{f_1(z)}$ and $\overline{f_2(z)}$
2. $\overline{f_1(-z)}$ and $\overline{f_2(-z)}$
3. $|f_1(z)|^2$ and $|f_2(z)|^2$
4. $f_1(z)\cdot f_2(z)$
5. $a_1\cdot f_1(z) + a_2\cdot f_2(z)$ for $a_1, a_2 > 0$

These properties can be easily proved using the definition of the Laplace transform (1).

Lemma 3

If $F(z)$ is co-positive definite, then $F(0)$ is real and $F(0) > 0$.

Proof

From (8), We let N =1, then $c_1\overline{c_1}F(x_1 + x_1) > 0$. Letting $\mathrm{x} = x_1 + x_1$, we have $|c_1|^2 \cdot F(x) > 0$. Hence $F(x)$ is real and $F(x) > 0$ for all real x and therefore $F(0)$ is real and $F(0) > 0$.

Lemma 4

If $F(z)$ is co-positive definite, then $F^{(2n)}(z)$ is also co-positive definite for non-negative integer n.

Proof

From the definition of the Laplace transform (1), taking derivative $2n$ times with respect to z, we have

$$\int_{-\infty}^{\infty} t^{2n} f(t) \cdot e^{-zt} dt$$

Since $f(t) > 0$ for all t and even, and therefore $t^{2n}f(t) > 0$ and $t^{2n}f(t)$ is even and hence co-positive definite.

Lemma 5

If y is fixed, say $\mathrm{y} = y_0$ and $\mathrm{z} = \mathrm{x} + \mathrm{i}y_0$, then any complex-valued co-positive definite function $F(z)$ is co-positive definite for x.

Proof

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n\overline{c_k}F(z_n + \overline{z_k}) =$$

$$= \int_{-\infty}^{\infty} f(t) \cdot \sum_{n=1}^{N}\sum_{k=1}^{N} c_n\overline{c_k}\, e^{-(z_n+\overline{z_k})t} dt$$

$$= \int_{-\infty}^{\infty} f(t) \cdot \sum_{n=1}^{N}\sum_{k=1}^{N} c_n\overline{c_k}\, e^{-(x_n+iy_0+x_k-iy_0)t} dt = \int_{-\infty}^{\infty} f(t) \cdot \sum_{n=1}^{N}\sum_{k=1}^{N} c_n\overline{c_k}\, e^{-(x_n+x_k)t} dt$$

$$= \int_{-\infty}^{\infty} f(t) \cdot \left|\sum_{n=1}^{N} c_n\, e^{-x_n t}\right|^2 dt > 0$$

This is clear since

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n\overline{c_k}F(z_n + \overline{z_k}) = \sum_{n=1}^{N}\sum_{k=1}^{N} c_n\overline{c_k}F(x_n + x_k) > 0$$

that is, y_0 is cancelled out and we have a real-valued co-positive definite function.

We note that this also is valid for $|\mathrm{F}(\mathrm{z})|^2$. If y is fixed, $|\mathrm{F}(x + iy_0)|^2$ is real-valued co-positive definite for x.

5. Expansion of $|F(z)|^2$

From (6), we have

$$\mathrm{F(z)} = \int_{-\infty}^{\infty} f(t) \cdot e^{-zt} dt = \sum_{n=0}^{\infty} a_{2n} \cdot z^{2n} \tag{9}$$

and $|F(z)|^2 = F(z) \cdot \overline{F(z)} = F(z) \cdot F(\bar{z}) = [\sum_{n=0}^{\infty} a_{2n} \cdot z^{2n}] \cdot [\sum_{n=0}^{\infty} a_{2n} \cdot \bar{z}^{2n}]$. The power series expansion of $[\sum_{n=0}^{\infty} a_{2n} \cdot z^{2n}] \cdot [\sum_{n=0}^{\infty} a_{2n} \cdot \bar{z}^{2n}]$ is

$$|F(z)|^2 = C_0 + \sum_{n=1}^{\infty} C_{2n} \cdot (z^{2n} + \bar{z}^{2n}) \tag{10}$$

where $C_0 = \sum_{k=0}^{\infty} a_{2k}^2 \, |z|^{4k}$, $C_{2n} = \sum_{k=0}^{\infty} a_{2k} \cdot a_{2k+2n} \, |z|^{4k}$. Since $a_{2k} \geq 0$, $C_0 > 0$ and $C_{2n} > 0$ for all n.

Since there are only even terms of x, we let

$$|F(z)|^2 = A_0 + \sum_{m=1}^{\infty} A_{2m} \cdot x^{2m} \tag{11}$$

To determine A_0 and A_{2m},

$$C_0 = \sum\nolimits_{k=0}^{\infty} a_{2k}^2 |z|^{4k} = \sum\nolimits_{k=0}^{\infty} a_{2k}^2 (x^2 + y^2)^{2k} = 2 \sum\nolimits_{j=0}^{\infty} \sum\nolimits_{k=j}^{\infty} a_{2k}^2 \binom{2k}{j} y^{4k-2j} x^{2j} \tag{12}$$

and since

$$(z^{2n} + \bar{z}^{2n}) = 2 \sum\nolimits_{p=0}^{n} (-1)^p \binom{2n}{2p} y^{2n-2p} x^{2p} \tag{13}$$

we have

$$\sum_{n=1}^{\infty} C_{2n} \cdot (z^{2n} + \bar{z}^{2n}) = \begin{cases} 2 \sum\nolimits_{k=0}^{\infty} \sum\nolimits_{n=1}^{\infty} (-1)^n \cdot y^{2n} \cdot C_{2n} & for \;\; p = 0 \\ 2 \sum\nolimits_{p=1}^{\infty} \sum\nolimits_{n=p}^{\infty} (-1)^n \cdot \binom{2n}{2n-2p} y^{2n-2p} \, x^{2p} \cdot C_{2n} & for \;\; p \geq 1 \end{cases} \tag{14}$$

and

$$C_{2n} = \sum\nolimits_{k=0}^{\infty} a_{2k} \cdot a_{2k+2n} \, |z|^{4k} = \sum\nolimits_{k=0}^{\infty} a_{2k} \cdot a_{2k+2n} \, (x^2 + y^2)^{2k} \tag{15}$$

By the Leibniz rule,

$$\frac{\partial^{2m}}{\partial x^{2m}} [(x^2 + y^2)^{2k} \cdot x^{2m}]_{\mathrm{x=0}} = \frac{\partial^{2m}}{\partial x^{2m}} [(x^2 + y^2)^{2k}]_{\mathrm{x=0}} = \left[\prod_{j=0}^{m-1} [(2m-j)(2k-j)] \right] \cdot y^{2(2k-m)} \tag{16}$$

where $\mathrm{m} \geq 1$ and $\mathrm{k} \geq \mathrm{m}$.

Taking derivatives (2m) times to Eq. (11) and Eq. (12), (14) and letting $\mathrm{x} = 0$, we have

$$A_{2m} = C_{2m}^1 + C_{2m}^2 + C_{2m}^3 \tag{17}$$

where

$$C_{2m}^1 = \sum\nolimits_{k=0}^{\infty} a_{2k}^2 \binom{2k}{2m} y^{4k-2m}$$

$$C_{2m}^2 = 2\sum_{k=m}^{\infty}\sum_{n=1}^{\infty}(-1)^n \cdot a_{2k} \cdot a_{2k+2n} \cdot \left[\prod_{j=0}^{m-1}(2m-j)(2k-j)\right] y^{2n} \cdot y^{2(2k-m)}$$

$$C_{2m}^3 = 2\sum_{n=m}^{\infty}(-1)^{n-m}\binom{2n}{2n-2m} y^{2n-2m}\sum_{k=m}^{\infty} a_{2k} \cdot a_{2k+2n}\left[\prod_{j=0}^{m-1}(2m-j)(2k-j)\right] \cdot y^{2(2k-m)}$$

and

$$A_0 = \sum_{k=0}^{\infty} a_{2k}^2 y^{4k} + 2\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-1)^n a_{2k}a_{2k+2n}y^{2n+4k} \tag{18}$$

If y is fixed, A_0 and A_{2m} are constants and $|F(z)|^2$ are a function of only x, that is, we have a function $|F(z)|^2$ which is lying on the horizontal line of $\mathrm{z} = x + iy_0$ where y_0 is a fixed value.

By Lemma 3 and letting $x = 0$, we note $A_0 > 0$. Likewise, from Eq. (11), taking derivative $2n$ times with respect to x and letting $x = 0$, we note $A_{2m} > 0$ for all $m \geq 1$ by Lemma 3 and 4.

Since all the coefficients of Eq. (11) positive, by Hardy-Littlewood, $|F(z)|^2$ is a strictly multiplicatively convex function on the horizontal line $\mathrm{z} = \mathrm{x} + y_0$ and in the interval of x $(0, \infty)$.

Moreover, since $|F(z)|^2$ is symmetric at $\mathrm{x} = 0$, $|F(z)|^2$ has a unique minimum at $\mathrm{x} = 0$. Further, since y_0 is arbitrary, we conclude the theorem:

Theorem

1. Let $F(z)$ be Laplace transform of a positive and even function, then if y is fixed, $|F(z)|^2$ is strictly multiplicatively convex for $0 \leq x < \infty$ on the line $\mathrm{z} = x + i\mathrm{y}_0$. If $F(z)$ is not entire, $|F(z)|^2$ is multiplicatively convex in the radius of convergence.
2. Since $|F(z)|^2$ is symmetric by iy-axis, $|F(z)|^2$ has a unique minimum at $x = 0$.
3. Since $|F(z)|^2$ has a unique minimum at $x = 0$, all zeros of $|F(z)|^2$ locate only at iy-axis. It cannot have zeros on other places since $|F(iy_0)|^2 \geq 0$ and $|F(x + i\mathrm{y}_0)|^2 > |F(i\mathrm{y}_0)|^2$ for all $x \neq 0$. Clearly, this is also valid for $|F(z)|$.
4. Since the zeros of $|F(z)|^2$ locate at iy-axis, the zeros of $F(z)$ locate only at iy-axis, if $F(z)$ has any zeros[2]. If $F(z)$ had zeros on other places than on iy-axis, $|F(z)|^2$ should have zeros on the same places, but $|F(z)|^2$ has zeros only on iy-axis as shown.

The theorem is valid only for the Laplace transform of functions that are even and positive for all t. It is not valid for non-even functions. However, the Laplace transform of the positive functions are co-positive definite and the power series can be found which is similar to (11) but contains odd powers. We do not have any information about the coefficients of odd powers but the coefficients of even powers should be positive if the power series converges.

[2] There are some functions whose Laplace transform does not have any zero. For example, $2\sqrt{\pi}e^{z^2}$ is the Laplace transform of $e^{-t^2/4}$, but $2\sqrt{\pi}e^{z^2}$ has neither poles nor zeros.

6. The positive definiteness

By replacing $\mathrm{z} \mapsto iz$ and for some function $f(t) > 0$, the Laplace transform will be

$$\mathrm{G}(\mathrm{z}) = \int_{-\infty}^{\infty} f(t) \cdot e^{-izt} dt \tag{19}$$

It can be shown that $\mathrm{G}(\mathrm{z})$ is complex-valued positive definite, that is

$$\sum_{n=1}^{N} \sum_{k=1}^{N} c_n \overline{c_k} G(z_n - \overline{z_k}) > 0 \tag{20}$$

which is the generalized Bochner's theorem.

The Properties of positive definiteness are similar to the co-positive definiteness. For example, if $\mathrm{G}(\mathrm{z})$ is positive definite, $(-1)^n G^{(2n)}(z)$ is also positive definite and $\mathrm{G}(0) > 0$ and therefore $\mathrm{G}(0)$ is real and so on.

If $f(t) > 0$ for all t and is even, we have:

$$\mathrm{G}(\mathrm{z}) = \int_{-\infty}^{\infty} f(t) \cdot e^{-izt} dt = 2 \int_{0}^{\infty} f(t) \cdot \cos(zt) dt = \int_{-\infty}^{\infty} f(t) \cdot \cos(zt) dt \tag{21}$$

Using the power series expansion and properties of the positive definite functions, it can be shown that $\mathrm{G}(\mathrm{z})$ from Eq. (21) has only real zeros. In fact, we do not need to prove it. Because of mapping $\mathrm{z} \mapsto iz$, $\mathrm{G}(\mathrm{z})$ is a rotated function by $\pi/2$ of $\mathrm{F}(\mathrm{z})$ in Eq. (9), and therefore, since $\mathrm{F}(\mathrm{z})$ in Eq. (9) has zeros only on the iy-axis, $\mathrm{G}(\mathrm{z})$ should have zeros only on x-axis, that is, only real zeros.

7. Behavior on a circle

We now consider $|F(z)|^2$ on a circle centered at the origin with radius r. Letting $z = r \cdot e^{i\theta}$, from Eq. (10) we have

$$\left|F(r \cdot e^{i\theta})\right|^2 = C_0 + 2 \sum_{n=1}^{\infty} C_{2n} \cdot r^{2n} \cdot \cos(2n\theta) \tag{22}$$

Since $|F(x)|^2$ is co-positive definite as shown, letting $x = r \cdot \cos(\theta)$ and if r is fixed, $|F(x)|^2$ is co-positive definite for $\cos(\theta)$, thus $\left|F(r \cdot e^{i\theta})\right|^2$ is also co-positive definite for $\cos(\theta)$. This can be easily proved using the definition of the Laplace transform Eq. (1). Caution must be taken. $\left|F(r \cdot e^{i\theta})\right|^2$ is co-positive definite for $\cos(\theta)$ itself, not for θ. If $\left|F(r \cdot e^{i\theta})\right|^2$ is co-positive definite for θ, it must be that $\frac{\partial^2}{\partial\theta^2}\left[\left|F(r \cdot e^{i\theta})\right|^2\right]_{\theta=0} > 0$. Taking derivative twice to Eq. (22) and letting $\theta = 0$, we have $-2 \cdot \sum_{n=1}^{\infty} (2n)^2 \cdot C_{2n} \cdot r^{2n}$. Since $C_{2n} > 0$ for all n, $-2 \cdot \sum_{n=1}^{\infty} (2n)^2 \cdot C_{2n} \cdot r^{2n} < 0$, thus $\left|F(r \cdot e^{i\theta})\right|^2$ is not co-positive definite for θ. In fact, $\left|F(r \cdot e^{i\theta})\right|^2$ is positive definite for θ because $\cos(2n\theta)$ is positive definite and $\left|F(r \cdot e^{i\theta})\right|^2$ is summing of $\cos(2n\theta)$ with positive coefficients, hence $\left|F(r \cdot e^{i\theta})\right|^2$ is positive definite for θ.

Letting $\alpha = \cos(\theta)$ and for fixed r, we have

$$|F(\alpha)|^2 = C_0 + 2\sum_{n=1}^{\infty} C_{2n} \cdot r^{2n} \cdot T_{2n}(\alpha) \tag{23}$$

where $T_{2n}(\alpha)$ denotes the Chebyshev polynomials of the first kind.

$T_n(\alpha)$ can be expressed as the sum of α, which is

$$T_n(\alpha) = \frac{1}{2}\sum_{k=0}^{\underline{n/2}} (-1)^k \frac{n}{n-k} \cdot \binom{n-k}{k} \cdot (2\alpha)^{n-2k} \tag{24}$$

where $\underline{n/2}$ denotes the floor function.

Letting $\mathrm{n} \mapsto 2\mathrm{n}$, we have

$$T_{2n}(\alpha) = \sum_{k=0}^{n} (-1)^k \frac{n}{2n-k} \cdot \binom{2n-k}{k} \cdot (2\alpha)^{2n-2k} \tag{25}$$

and letting $\mathrm{m} = \mathrm{n} - \mathrm{k}$

$$T_{2n}(\alpha) = \sum_{m=0}^{n} (-1)^{n-m} \frac{n}{n+m} \cdot \binom{n+m}{n-m} \cdot (2\alpha)^{2m} \tag{26}$$

and finally we have

$$|F(\alpha)|^2 = C_0 + 2\sum_{n=1}^{\infty} C_{2n} \cdot r^{2n} \cdot \sum_{m=0}^{n} (-1)^{n-m} \frac{n}{n+m} \cdot \binom{n+m}{n-m} \cdot 2^{2m} \cdot \alpha^{2m} \tag{27}$$

From Eq. (27), if $m = 0$, then we get

$$C_0 + 2\sum_{n=1}^{\infty} (-1)^n \cdot C_{2n} \cdot r^{2n} \tag{228}$$

which is the constant part of $|F(\alpha)|^2$.

With the results, we have

$$|F(\alpha)|^2 = C_0 + 2\sum_{n=1}^{\infty} (-1)^n \cdot C_{2n} \cdot r^{2n} + 2\sum_{m=1}^{\infty}\sum_{n=m}^{\infty} (-1)^{n-m} \frac{n}{n+m} \cdot \binom{n+m}{n-m} C_{2n} \cdot r^{2n} \cdot 2^{2m} \cdot \alpha^{2m} \tag{239}$$

which can be written

$$|F(\alpha)|^2 = B_0 + B_2\alpha^2 + \cdots = B_0 + \sum_{m=1}^{\infty} B_{2m} \cdot \alpha^{2m} \tag{30}$$

where

$$B_0 = C_0 + 2\sum_{n=1}^{\infty} (-1)^n \cdot C_{2n} \cdot r^{2n}$$

and

$$B_{2m} = \sum_{n=m}^{\infty} (-1)^{n-m} \frac{n}{n+m} \cdot \binom{n+m}{n-m} C_{2n} \cdot r^{2n} \cdot 2^{2m} \cdot$$

Since $|F(\alpha)|^2$ is co-positive definite, we have

$$|F(0)|^2 > 0 \text{ and } \frac{d^{2m}}{d\alpha^{2m}}[|F(\alpha)|^2]_{\alpha=0} > 0$$

thus

$$B_0 > 0 \text{ and } B_{2m} > 0 \text{ for } m = 1,2,\cdots$$

Since $\alpha = \cos(\theta)$, from Eq. (30)

$$|F(\alpha)|^2 = B_0 + \sum_{m=1}^{\infty} B_{2m} \cdot \cos^{2m}(\theta) \tag{31}$$

and

$$\frac{d}{d\theta}|F(\alpha)|^2 = -2\sin(\theta)\sum_{m=1}^{\infty} m \cdot B_{2m} \cdot \cos^{2m-1}(\theta) = -\sin(2\theta)\sum_{m=1}^{\infty} m \cdot B_{2m} \cdot \cos^{2m-2}(\theta) \tag{32}$$

Since

$$\sum_{m=1}^{\infty} m \cdot B_{2m} \cdot \cos^{2m-2}(\theta) = B_2 + \sum_{m=2}^{\infty} m \cdot B_{2m} \cdot \cos^{2m-2}(\theta)$$

$\sum_{m=1}^{\infty} m \cdot B_{2m} \cdot \cos^{2m-2}(\theta)$ cannot be zero and therefore $\sum_{m=1}^{\infty} m \cdot B_{2m} \cdot \cos^{2m-2}(\theta)$ is strictly positive, thus $-\sin(2\theta)$ determines the sign of $\frac{d}{d\theta}|F(\alpha)|^2$.

We note that $\sin(2\theta) = 0$ when $\theta = 0$ and $\theta = \pi/2$ as expected. Moreover, $-\sin(2\theta) < 0$ in the interval $0 < \theta < \pi/2$, and therefore $|F(\alpha)|^2$ is decreasing in the interval. Since $|F(\alpha)|^2 \geq 0$, $|F(\alpha)|^2$ has a unique minimum at $\theta = \pi/2$ and $\theta = -\pi/2$ because $|F(\alpha)|^2$ has the same value at $\theta, -\theta$, $\pi - \theta$ and $\theta - \pi$.

Since the radius r can be chosen arbitrarily, the zeros of $F(\alpha)$ locate only $\theta = \pi/2$ and $= -\pi/2$, that is, iy-axis which we have proved. In fact, it can be shown that if $|F(\alpha)|^2$ has a unique minimum at $= \pi/2$, $|F(z)|^2$ is strictly multiplicatively convex for x on the line $z = x + iy_0$ where $y_0 = r$.

8. Proof of the Riemann hypothesis

The Riemann zeta function $\zeta(s)$ is defined as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \tag{33}$$

where $s = \sigma + i\omega$.

The functional equation of Riemann xi-function $\xi(s)$ is

$$\xi(s) = \xi(1 - s) \tag{34}$$

where

$$\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}} s(s-1)\Gamma(s/2)\zeta(s) \tag{35}$$

which is symmetric at $\sigma = \frac{1}{2}$ and therefore $\xi\left(\frac{1}{2} + i\omega\right)$ is real-valued.

Riemann showed that

$$\xi(s) = \frac{1}{2} + \frac{1}{2}\int_1^{\infty}\left[\sum_{n=1}^{\infty} e^{-n^2\pi t}\right] \cdot \frac{s(s-1)}{t}\left(t^{s/2} + t^{(1-s)/2}\right)dt \tag{36}$$

The Jacobi Theta function can be expanded using the Poisson summation formula and the Mellin transform, and finally we have

$$\xi(s)=\int_{-\infty}^{\infty}\varphi(t)\cdot e^{\left(s-\frac{1}{2}\right)t}dt \tag{24}$$

where

$$\varphi(t)=2\sum_{n=1}^{\infty}e^{-\pi n^2 e^{2t}}\left(2\pi^2 n^4 e^{\frac{9}{2}t}-3\pi n^2 e^{\frac{5}{2}t}\right) \tag{38}$$

It can be shown that $\varphi(t)>0$ for all t and an even function. From (37), the power series expansion is

$$\xi(s)=\sum_{n=0}^{\infty}h_{2n}\left(s-\frac{1}{2}\right)^{2n} \tag{39}$$

Letting $z\mapsto s-\frac{1}{2}$, that is, shifted by $\frac{1}{2}$ and therefore the zeros now locate on the stripe of $-\frac{1}{2}<x<\frac{1}{2}$ and we have

$$\Phi(z)=\int_{-\infty}^{\infty}\varphi(t)\cdot e^{zt}dt=\int_{-\infty}^{\infty}\varphi(t)\cdot e^{-zt}dt \tag{25}$$

and the power series expansion

$$\Phi(z)=\sum_{n=0}^{\infty}h_{2n}\cdot z^{2n} \tag{41}$$

which is the Laplace transform of the positive and even function $\varphi(t)$. We showed that all zeros of the Laplace transform of a positive and even functions locate at iy-axis. Hence all the zeros of $\xi(s)$ must locate at $\sigma=\frac{1}{2}$ and therefore $\zeta(s)$ too, which means the Riemann hypothesis is true.

Riemann suggested a function named “big xi-function” $\Xi(z)$, which is defined

$$\Xi(z)=2\int_{0}^{\infty}\varphi(t)\cdot\cos(zt)\cdot dt=\int_{-\infty}^{\infty}\varphi(t)\cdot\cos(zt)\cdot dt \tag{42}$$

and if $\Xi(z)$ has only real zeros, the Riemann hypothesis is true. This function is nothing but a positive definite function in Eq. (21) and we showed that these kind of functions can have only real zeros and therefore the Riemann hypothesis is true.

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