Conformal Kerr-de Sitter Gravity

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1. Introduction

$\ldots = $ "a mess."

That was New Zealand mathematician Roy Kerr at the blackboard one day, perhaps reflecting on the exasperatingly difficult mathematics of his exact solution for a rotating black hole.

For over one hundred years, the Schwarzschild solution to Einstein's gravitational field equations has been a uniquely valuable tool in the study planetary motion, black holes and even basic stellar structure. Karl Schwarzschild worked out the solution just months after Einstein's November 1915 announcement of the theory of general relativity, but it was already viewed as merely an idealization, since it holds only for non-rotating masses. All real cosmological bodies rotate but, unlike Schwarzschild's comparatively simple derivation, an exact solution for a spinning mass evaded physicists for nearly fifty years. Kerr cracked the problem in 1963, but the solution was so complicated that the importance of his discovery was not fully recognized for a decade. But like Dirac's relativistic electron equation, the Kerr equation has become a cornerstone of modern physics.

These days every undergraduate physics student can derive the Schwarzschild metric, but replicating Kerr's solution remains a tremendous challenge. And that's just for a single rotating body—the recent astronomical discovery of merging, rotating binary black holes, neutron stars and their attendant gravitational radiation required extensive computational effort on high-speed computers using numerical software rivaling Kerr's work in terms of sheer mathematical complexity.

Kerr's discovery opened up an entire new world in cosmology, introducing or confirming bizarre concepts such as inertial frame dragging, the Penrose energy-extraction process, the possibility of naked singularities and even wormholes to parallel universes. But its major contribution has been the ability to understand the physics of rotating stars and black holes, the only physically realistic massive bodies in the universe.

2. The Kerr Solution

Like the Schwarzschild solution, the Kerr metric is a unique solution to the free-space Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{2.1}$$

but includes terms in the metric associated with the angular momentum J of the central rotating mass M. Kerr's solution, expressed in the familiar differential coordinates $cdt, r, \theta, d\phi$ is given by

$$ds^2 = \left(1 - \frac{2mr}{\Sigma}\right) c^2 dt^2 - \frac{\Sigma}{\Delta} dr^2 - \Sigma\, d\theta^2 - \left(r^2 + a^2 + \frac{2mra^2}{\Sigma} \sin^2\theta\right) \sin^2\theta\, d\phi^2 + \frac{4mra}{\Sigma} \sin^2\theta\, cdtd\phi \tag{2.2}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2mr + a^2$ (Boyer-Lindquist coordinates), $a = J/m$ is the rotation parameter, and $m = GM/c^2$. Note the single off-diagonal term g_{03}, which produces the effect of rotation about the ϕ plane. The spacetime around a rotating mass thus describes an oblate spheroid, symmetric about the z-axis, as expected. For $a = 0$, the the Kerr metric reduces immediately to the Schwarzschild solution.

The Kerr problem is complicated by the fact that no fewer than twenty Christoffel symbols must be computed, many of which are functions of both the radial coordinate r and the azimuthal angle θ, while there are five distinct $g_{\mu\nu}$ terms displaying similar coordinate dependence. A brute force approach quickly appears hopeless, although the basic strategem is straightforward. It is a tribute to Kerr's brilliance that he was able to deduce the solution without the benefit of computers, which today are absolutely necessary in utilizing the Kerr solution in cosmological applications.

The Kerr metric has many fascinating properties, but in the interest of brevity we will not describe them here. The student is encouraged to consult the references for details.

3. Conformal Invariance

> *There is one strong reason in support of [Weyl's conformal theory]. It appears as one of the fundamental principles of Nature that the equations expressing basic laws should be invariant under the widest possible group of transformations.* — Dirac, 1973

The Einstein equations may be derived by performing a variation of the metric tensor $g^{\mu\nu}$ for the free-space Einstein-Hilbert action

$$S_{EH} = \int \sqrt{-g}\, R\, d^4x \tag{3.1}$$

where $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar. The variation leads to the usual equations of motion

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 0$$

While the Einstein-Hilbert action is Lorentz and coordinate invariant, it is not invariant with respect to a local conformal (or scale) transformation defined by $g_{\mu\nu} \to e^{\pi} g_{\mu\nu}$, where $\pi(x)$ is an arbitrary scalar field. Conformal invariance guarantees that physical laws are not violated under an arbitrary local redefinition of length or energy. It is closely related to the *gauge invariance* of electrodynamics, where an arbitrary addition of a scalar gradient to the four-vector A_μ has no effect on electric or magnetic fields. While conformal symmetry is an extension of the Poincaré group of transformations, its relevance to general relativity was first considered by the German mathematical physicist Hermann Weyl in 1918. Its relevance in cosmology is a topic of much research today, although it is still not known if Nature demands conformal invariance in her physical laws.

The Einstein-Hilbert action can be made conformally invariant by appending various suitable scalar and vector fields to R in the action Lagrangian that force it to be invariant, but such schemes can be avoided by considering the Riemannian Lagrangian

$$L = \sqrt{-g}\left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2\right) \tag{3.2}$$

which, although quadratic in the Ricci tensor and scalar, is fully conformally invariant (see Ref. 9 for details). While Mannheim and Kazanas found an exact solution to the associated equations of motion for this Lagrangian, they are very complicated and somewhat questionable in terms of cosmological relevance. This prompts the question of whether a simpler action exists that serves the same purpose. Following the early work of Weyl, let us consider instead the expression

$$S = \int \sqrt{-g}\, R^2\, d^4x \tag{3.3}$$

It is an easy matter to show (again, see Ref. 9) that under a conformal variation this reduces to

$$\frac{\delta S}{\delta g^{\mu\nu}} = -6 \int \sqrt{-g}\, g^{\mu\nu}\, \partial_\nu R\, \partial_\mu \pi\, d^4x = 0 \tag{3.4}$$

The associated equations of motion are

$$R\left(R_{\mu\nu} - \frac{1}{4}\, g_{\mu\nu} R\right) = 0 \tag{3.5}$$

Clearly, taking $R = 0$ or $\partial_\nu R = 0$ satisfies (3.4), but (3.5) is non-trivial only for $R =$ non-zero constant. Dividing this out, we are then left with the traceless expression

$$R_{\mu\nu} - \frac{1}{4}\, g_{\mu\nu} R = 0 \tag{3.6}$$

It is a simple matter to show that the solution to these equations is

$$g_{00} = -1/g_{11} = 1 - \frac{2m}{r} - \frac{1}{12} R r^2, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta$$

which was discovered long ago in conjunction with the Einstein equations having a non-zero cosmological constant Λ,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \tag{3.7}$$

Taking the trace of (3.7), we have again the expression in (3.6). Consequently, we see that a constant Ricci scalar R plays the role of the cosmological constant.

Thus, $R =$ constant not only results in a traceless set of motion equations, but also produces a simple, conformally invariant theory. And by dividing out this constant in (3.5), we're left with a set of equations that are of only second order in the metric tensor and its derivatives, a desirable property that is lacking in the Mannheim-Kazanas solution.

4. Conformal Kerr-de Sitter Gravity

For a non-vanishing cosmoligical constant Λ in an otherwise empty universe, (3.7) represents de Sitter spacetime. Cosmologists strongly believe that such a spacetime provides an accurate description for a future universe in which accelerated expansion has left a universe largely devoid of matter.

We would now like to derive the Kerr metric for such (3.6), since we now know it to be conformally invariant for constant R. Ordinarily, this would be a tremendous undertaking, but we can take advantage of the fact that the Kerr metric for (4) has already been worked out by Carter, who investigated the Einstein equations using the energy-momentum tensor $T_{\mu\nu} = \Lambda g_{\mu\nu}$, or

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} = -\frac{8\pi G}{c^4} \Lambda g_{\mu\nu} \tag{4.1}$$

Again, contraction with $g^{\mu\nu}$ reduces this to (3.6).

We now have only to write down the solution Carter derived in 1973 which, using the set of coordinates employed by Akcay and Matzner, is

$$ds^2 = \frac{c^2 dt^2}{\Sigma_R} \left(\Delta_r - a^2 \sin^2 \theta \Delta_R \right) - \Sigma \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_R} \right)$$

$$- \frac{\sin^2 \theta d\phi^2}{\Sigma_R} \left[\left(r^2 + a^2 \right) \Delta_R - a^2 \sin^2 \theta \Delta_r \right] + \frac{2a \sin^2 \theta \, c dt d\phi}{\Sigma_R} \left[\left(r^2 + a^2 \right) \Delta_R - \Delta_r \right] \tag{4.2}$$

where

$$\Delta_r = \Delta - \frac{1}{12} R r^2 \left(r^2 + a^2 \right), \quad \Delta_R = 1 + \frac{1}{12} R a^2 \cos^2 \theta, \quad \Sigma_R = \Sigma \left(1 + \frac{1}{12} R a^2 \right)^2 \tag{4.3}$$

Admittedly, this is an even bigger "mess" than the one Kerr originally arrived at, but it has the advantage of being conformally invariant. And while we have assumed that $R \neq 0$, there are no other constraints on the Ricci scalar, so we are free to set it to some small, ignorable constant. We thus recover the orginal Kerr spacetime (2.2), which is essentially conformal as it stands.

5. Comments

While the Kerr solution of the free-space Einstein field equations is complicated enough, it is heartening that the cosmological constant can be incorporated into the solution with little additional effort. It is similarly heartening to note that its inclusion provides a straightforward way of bringing conformal theory into the formalism.

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