An extended version of the Natario warp drive equation based in the original $3 + 1$ ADM formalism which encompasses accelerations and variable velocities

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December 4, 2017

Abstract

Warp Drives are solutions of the Einstein Field Equations that allows superluminal travel within the framework of General Relativity. There are at the present moment two known solutions: The Alcubierre warp drive discovered in 1994 and the Natario warp drive discovered in 2001. However the major drawback concerning warp drives is the huge amount of negative energy density able to sustain the warp bubble.In order to perform an interstellar space travel to a "nearby" star at 20 light-years away in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!. With the correct form of the shape function the Natario warp drive can overcome this obstacle at least in theory.Other drawbacks that affects the warp drive geometry are the collisions with hazardous interstellar matter that will unavoidably occurs when a ship travels at superluminal speeds and the problem of the Horizons(causally disconnected portions of spacetime).The geometrical features of the Natario warp drive are the required ones to overcome these obstacles also at least in theory.However both the Alcubierre or Natario warp drive spacetimes always have a constant speed in the internal structure of their equations which means to say that these warp drives always travel with a constant speed.But a real warp drive must accelerate from zero to a superluminal speed of about 200 times faster than light in the beginning of an interstellar journey and de-accelerate again to zero in the end of the journey.In this work we expand the Natario vector introducing the coordinate time as a new Canonical Basis for the Hodge star and we introduce an extended Natario warp drive equation which encompasses accelerations.

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1 Introduction:

The Warp Drive as a solution of the Einstein field equations of General Relativity that allows superluminal travel appeared first in 1994 due to the work of Alcubierre.([1]) The warp drive as conceived by Alcubierre worked with an expansion of the spacetime behind an object and contraction of the spacetime in front.The departure point is being moved away from the object and the destination point is being moved closer to the object.The object do not moves at all[1].It remains at the rest inside the so called warp bubble but an external observer would see the object passing by him at superluminal speeds(pg 8 in [1])(pg 1 in [2]).

Later on in 2001 another warp drive appeared due to the work of Natario.([2]).This do not expands or contracts spacetime but deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [2]). Imagine the object being a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium whose walls do not expand or contract. An observer in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.

However there are 3 major drawbacks that compromises the warp drive physical integrity as a viable tool for superluminal interstellar travel.

The first drawback is the quest of large negative energy requirements enough to sustain the warp bubble. In order to travel to a "nearby" star at 20 light-years at superluminal speeds in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!(see [7],[8] and [9]).

Another drawback that affects the warp drive is the quest of the interstellar navigation:Interstellar space is not empty and from a real point of view a ship at superluminal speeds would impact asteroids,comets,interstellar space dust and photons.(see [5],[7] and [8]).

The last drawback raised against the warp drive is the fact that inside the warp bubble an astronaut cannot send signals with the speed of the light to control the front of the bubble because an Horizon(causally disconnected portion of spacetime)is established between the astronaut and the warp bubble.(see [5],[7] and [8]).

We can demonstrate that the Natario warp drive can "easily" overcome these obstacles as a valid candidate for superluminal interstellar travel(see [7],[8] and [9]).

In this work we cover only the Natario warp drive and we avoid comparisons between the differences of the models proposed by Alcubierre and Natario since these differences were already deeply covered by the existing available literature.(see [5],[6] and [7])However we use the Alcubierre shape function to define its Natario counterpart.

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[1]do not violates Relativity

Alcubierre([12]) used the so-called 3+1 Arnowitt-Dresner-Misner(ADM) formalism using the approach of Misner-Thorne-Wheeler(MTW)([11]) to develop his warp drive theory.As a matter of fact the first equation in his warp drive paper is derived precisely from the original $3+1$ ADM formalism(see eq 2.2.4 pgs [67(b)],[82(a)] in [12], see also eq 1 pg 3 in [1])[2][3] and we have strong reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to develop the Natario warp drive spacetime.

The Natario warp drive equation that obeys the $3+1$ ADM formalism is given below:

$$ds^2 = (1 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{1}$$

However some important things must be outlined in both the Alcubierre or Natario warp drive spacetimes:

- 1)-The warp drives as proposed by Alcubierre or Natario always have a constant speed vs.They do not accelerate or de-accelerate and travel always with a constant speed.But a real warp drive must "know" how to accelerate for example from 0 to a speed of 200 times faster than light in the beginning of an interstellar journey and in the end of the journey it must de-accelerate again to 0 in the arrival at the destination point which means to say of course a distant star.
- 2)-The warp drives as proposed by Alcubierre or Natario always have a constant speed vs raised to the square in their equations for the negative energy density.An accelerating warp drive probably must have the terms of variable velocities or accelerations included in the expression for the negative energy density since this energy is responsible for the generation of the warp drive spacetime.

Since the Natario vector is the generator of the Natario warp drive spacetime metric in this work we expand the original Natario vector including the coordinate time as a new Canonical Basis for the Hodge star generating an expanded Natario vector and an extended Natario warp drive spacetime metric which encompasses accelerations and variable velocities.Our proposed extended Natario warp drive metric with variable velocity vs due to a constant acceleration a is given by the following equation:

$$ds^2 = (1 - 2X_t + X_t X^t - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{2}$$

Note that in this equation a new set of contravariant and covariant components X^t and X_t appears because in this case as the velocity vs changes its value as times goes by due to a constant acceleration a this affects the whole spacetime geometry.

Two important things must be outlined by now:

- 1)-The Natario shape function used in the equation with constant speed is valid also in the equation with variable speed.
- 2)-Both equations satisfies the Natario criteria for a warp drive spacetime.

[2]see also Appendix E

[3]see the Remarks section on our system to quote pages in bibliographic references

In this work we present the new extended equation for the Natario warp drive spacetime which encompasses accelerations and variable speeds using also the ADM formalism and we arrive at the conclusion that the new equation is also a valid solution for the warp drive spacetime according to the Natario criteria.

For the study of the original ADM formalism we use the approaches of MTW([11]) and Alcubierre([12]) and we adopt the Alcubierre convention for notation of equations and scripts.

We adopt here the Geometrized system of units in which $c = G = 1$ for geometric purposes and the International System of units for energetic purposes.

This work is organized as follows:

- Section 2)-Introduces the Natario warp drive continuous shape function able to low the negative energy density requirements when a ship travels with a speed of 200 times faster than light.
  The negative energy density for such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!.

- Section 3)-presents the original equation for the Natario warp drive spacetime with a constant velocity vs in the $3+1$ ADM formalism in a rigorous mathematical fashion.We recommend the study of the Appendices A and E at the end of the work in order to fully understand the mathematical demonstrations.

- Section 4)-presents the extended equation for the Natario warp drive spacetime with a variable velocity vs and a constant acceleration a in the $3+1$ ADM formalism in a rigorous mathematical fashion.We recommend the study of the Appendices B,C and F at the end of the work in order to fully understand the mathematical demonstrations.

- Section 5)-compares both the original and the extended equations and we point out the fact that the shape function used in one equation is also valid in the other and since the derivatives of first or second order low the negative energy density requirements in the first equation these derivatives may perhaps be able to low the same requirements in the second equation.

This work must be regarded as a companion work to our works in [16] and in [17].

2 The Natario warp drive continuous shape function

Introducing here $f(rs)$ as the Alcubierre shape function that defines the Alcubierre warp drive spacetime we can construct the Natario shape function $n(rs)$ that defines the Natario warp drive spacetime using its Alcubierre counterpart.Below is presented the equation of the Alcubierre shape function.[4].

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{3}$$

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \tag{4}$$

According with Alcubierre any function $f(rs)$ that gives 1 inside the bubble and 0 outside the bubble while being $1 > f(rs) > 0$ in the Alcubierre warped region is a valid shape function for the Alcubierre warp drive.(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2]).

In the Alcubierre shape function xs is the center of the warp bubble where the ship resides. R is the radius of the warp bubble and @ is the Alcubierre parameter related to the thickness.According to Alcubierre these can have arbitrary values.We outline here the fact that according to pg 4 in [1] the parameter @ can have arbitrary values.rs is the path of the so-called Eulerian observer that starts at the center of the bubble $xs = R = rs = 0$ and ends up outside the warp bubble $rs > R$.

According to Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive.

The Natario warp drive continuous shape function can be defined by:

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{5}$$

$$n(rs) = \frac{1}{2}[1 - [\frac{1}{2}[1 - tanh[@(rs - R)]]]] \tag{6}$$

This shape function gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region.

Note that the Alcubierre shape function is being used to define its Natario shape function counterpart.

For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

[4] $tanh[@(rs + R)] = 1$,$tanh(@R) = 1$ for very high values of the Alcubierre thickness parameter $@ >> |R|$

Another Natario warp drive valid shape function can be given by:

$$n(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{7}$$

Its derivative square is :

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{8}$$

The shape function above also gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region(see pg 5 in [2]).

Note that like in the previous case the Alcubierre shape function is being used to define its Natario shape function counterpart. The term WF in the Natario shape function is dimensionless too:it is the warp factor.It is important to outline that the warp factor $WF >> |R|$ is much greater than the modulus of the bubble radius.

For the second Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

- Numerical plot for the second shape function with $@ = 50000$ $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99970000000E+001$	1	0	$2,650396620740E-251$	0
$9,99980000000E+001$	1	0	$1,915169647489E-164$	0
$9,99990000000E+001$	1	0	$1,383896564748E-077$	0
$1,00000000000E+002$	0,5	0,5	$6,250000000000E+008$	$3,872591914849E-103$
$1,00001000000E+002$	0	0,5	$1,383896486082E-077$	0
$1,00002000000E+002$	0	0,5	$1,915169538624E-164$	0
$1,00003000000E+002$	0	0,5	$2,650396470082E-251$	0

- Numerical plot for the second shape function with $@ = 75000$ $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99980000000E+001$	1	0	$5,963392481410E-251$	0
$9,99990000000E+001$	1	0	$1,158345097767E-120$	0
$1,00000000000E+002$	0,5	0,5	$1,406250000000E+009$	$8,713331808411E-103$
$1,00001000000E+002$	0	0,5	$1,158344999000E-120$	0
$1,00002000000E+002$	0	0,5	$5,963391972940E-251$	0

- Numerical plot for the second shape function with $@ = 100000$ $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99990000000E+001$	1	0	$7,660678807684E-164$	0
$1,00000000000E+002$	0,5	0,5	$2,500000000000E+009$	$1,549036765940E-102$
$1,00001000000E+002$	0	0,5	$7,660677936765E-164$	0

The plots in the previous page demonstrate the important role of the thickness parameter @ in the warp bubble geometry wether in both Alcubierre or Natario warp drive spacetimes.For a bubble of 100 meters radius $R = 100$ the regions where $1 > f(rs) > 0$(Alcubierre warped region) and $0 < n(rs) < \frac{1}{2}$(Natario warped region) becomes thicker or thinner as @ becomes higher.

Then the geometric position where both Alcubierre and Natario warped regions begins with respect to R the bubble radius is $rs = R - \epsilon < R$ and the geometric position where both Alcubierre and Natario warped regions ends with respect to R the bubble radius is $rs = R + \epsilon > R$

As large as @ becomes as smaller ϵ becomes too.

Note from the plots of the previous page that we really have two warped regions:

- 1)-The geometrized warped region where $1 > f(rs) > 0$(Alcubierre warped region) and $0 < n(rs) < \frac{1}{2}$ (Natario warped region).
- 2)-The energized warped region where the derivative squares of both Alcubierre and Natario shape functions are not zero.

The parameter @ affects both energized warped regions wether in Alcubierre or Natario cases but is more visible for the Alcubierre shape function because the warp factor WF in the Natario shape functions squeezes the energized warped region into a very small thickness.

The negative energy density for the Natario warp drive is given by(see pg 5 in [2])

$$\rho = T_{\mu\nu} u^{\mu} u^{\nu} = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3(n'(rs))^2 \cos^2\theta + \left(n'(rs) + \frac{r}{2} n''(rs) \right)^2 \sin^2\theta \right] \tag{9}$$

Converting from the Geometrized System of Units to the International System we should expect for the following expression(see Appendix G):

$$\rho = -\frac{c^2}{G} \frac{vs^2}{8\pi} \left[3(n'(rs))^2 \cos^2\theta + \left(n'(rs) + \frac{rs}{2} n''(rs) \right)^2 \sin^2\theta \right]. \tag{10}$$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for(see Appendix D):

$$\rho = T_{\mu\nu} u^{\mu} u^{\nu} = -\frac{c^2}{G} \frac{v_s^2}{8\pi} \left[3(n'(rs))^2 (\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2} n''(rs) \right)^2 (\frac{y}{rs})^2 \right] \tag{11}$$

In the equatorial plane($1+1$ dimensional spacetime with $rs = x - xs$,$y = 0$ and center of the bubble $xs = 0$):

$$\rho = T_{\mu\nu} u^{\mu} u^{\nu} = -\frac{c^2}{G} \frac{v_s^2}{8\pi} \left[3(n'(rs))^2 \right] \tag{12}$$

Note that in the above expressions the warp drive speed vs appears raised to a power of 2. Considering our Natario warp drive moving with $vs = 200$ which means to say 200 times light speed in order to make a round trip from Earth to a nearby star at 20 light-years away in a reasonable amount of time(in months not in years) we would get in the expression of the negative energy the factor $c^2 = (3 \times 10^8)^2 = 9 \times 10^{16}$ being divided by $6,67 \times 10^{-11}$ giving $1,35 \times 10^{27}$ and this is multiplied by $(6 \times 10^{10})^2 = 36 \times 10^{20}$ coming from the term $vs = 200$ giving $1,35 \times 10^{27} \times 36 \times 10^{20} = 1,35 \times 10^{27} \times 3,6 \times 10^{21} = 4,86 \times 10^{48}$!!!

A number with 48 zeros!!!The planet Earth have a mass[5] of about $6 \times 10^{24} kg$

This term is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!or better:The amount of negative energy density needed to sustain a warp bubble at a speed of 200 times faster than light requires the magnitude of the masses of 1.000.000.000.000.000.000.000.000 planet Earths!!!

Note that if the negative energy density is proportional to 10^{48} this would render the warp drive impossible but fortunately the square derivative of the Natario shape function possesses values of 10^{-102} ameliorating the factor 10^{48} making the warp drive negative energy density more "affordable". For a detailed study of the derivatives of first and second order of the Natario shape function $n(rs)$ see pgs 10 to 41 in [18]

[5] see Wikipedia:The free Encyclopedia

3 The equation of the Natario warp drive spacetime metric with a constant speed vs in the original $3+1$ ADM formalism

The equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:(see Appendix E for details)

$$ds^2 = (1 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{13}$$

The equation of the Natario vector nX(pg 2 and 5 in [2]) is given by:

$$nX = X^{rs}drs + X^\theta rsd\theta \tag{14}$$

With the contravariant shift vector components X^{rs} and X^θ given by:(see pg 5 in [2])(see also Appendix A for details)

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{15}$$

$$X^\theta = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{16}$$

The covariant shift vector components X_{rs} and X_θ are given by:

$$X_{rs} = 2v_s n(rs)\cos\theta \tag{17}$$

$$X_\theta = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{18}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We must demonstrate that the Natario warp drive equation given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x-axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

In a $1+1$ spacetime the equatorial plane we get¿:

$$ds^2 = (1 - X_{rs}X^{rs})dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{19}$$

But since $X_{rs} = X^{rs}$ the equation can be written as given below:

$$ds^2 = (1 - [X^{rs}]^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{20}$$

Examining the Natario warp drive equation in a $1+1$ spacetime:

$$ds^2 = (1 - [X^{rs}]^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{21}$$

The contravariant shift vector component X^{rs} is then:

$$X^{rs} = 2v_s n(rs) \tag{22}$$

Remember that Natario(pg 4 in [2]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $X^{rs} = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $X^{rs} = vs$ and this illustrates the Natario definition for a warp drive spacetime.

4 The equation of the Natario warp drive spacetime metric with a variable speed vs due to a constant acceleration a in the original $3+1$ ADM formalism

The equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:(see Appendix F for details)

$$ds^2 = (1 - 2X_t + X_tX^t - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2d\theta^2 \tag{23}$$

The equation of the Natario vector nX is given by:

$$nX = X^tdt + X^{rs}drs + X^\theta rsd\theta \tag{24}$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by(see Appendices B and C):

$$X^t = 2n(rs)rscos\theta a \tag{25}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{26}$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \tag{27}$$

The covariant shift vector components X_t,X_{rs} and X_θ are given by:

$$X_t = 2n(rs)rscos\theta a \tag{28}$$

$$X_{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{29}$$

$$X_\theta = -2n(rs)at[2n(rs) + rsn'(rs)]rs^2\sin\theta \tag{30}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We must demonstrate that the Natario warp drive equation given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx + xdvs$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x-axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

In a $1+1$ spacetime the equatorial plane we get¿:

$$ds^2 = (1 - 2X_t + X_t X^t - X_{rs} X^{rs})dt^2 + 2(X_{rs} drs)dt - drs^2 \tag{31}$$

But since $X_t = X^t$ and $X_{rs} = X^{rs}$ the equation can be written as given below:

$$ds^2 = (1 - 2X_t + (X_t)^2 - (X_{rs})^2)dt^2 + 2(X_{rs} drs)dt - drs^2 \tag{32}$$

$$X_t = 2n(rs)rsa \tag{33}$$

$$X_{rs} = 2[2n(rs)^2 + rsn'(rs)]at \tag{34}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2n(rs)at \tag{35}$$

Remember that Natario(pg 4 in [2]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $vs = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $n(rs)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $n'(rs)$ of the Natario shape function $n(rs)$ is zero[6] and the covariant shift vector $X_{rs} = 2[2n(rs)^2]at$ with $X_{rs} = 0$ inside the bubble and $X_{rs} = 2[2n(rs)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.

[6] except in the neighborhoods of the bubble radius.See Section 2

5 Differences between both equations with constant or variable velocity vs in the original $3+1$ ADM formalism for the Natario warp drive spacetime

The equation in $3+1$ ADM formalism for the Natario warp drive spacetime for a variable velocity vs is given by:(see Appendices B,C and F for details)

$$ds^2 = (1 - 2X_t + X_tX^t - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2d\theta^2 \tag{36}$$

The equation in $3+1$ ADM formalism for the Natario warp drive spacetime for a constant velocity vs is given by:(see Appendices A and E for details)

$$ds^2 = (1 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2d\theta^2 \tag{37}$$

The Natario vector for the equation with variable velocity vs is given by:

$$nX = X^tdt + X^{rs}drs + X^\theta rsd\theta \tag{38}$$

The Natario vector for the equation with constant velocity vs is given by:

$$nX = X^{rs}drs + X^\theta rsd\theta \tag{39}$$

The contravariant and covariant components in the Natario vector for the equation with variable velocity vs are given by:

$$X^t = 2n(rs)rscos\theta a | X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta | X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \tag{40}$$

$$X_t = 2n(rs)rscos\theta a | X_{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta | X_\theta = -2n(rs)at[2n(rs) + rsn'(rs)]rs^2\sin\theta \tag{41}$$

The contravariant and covariant components in the Natario vector for the equation with constant velocity vs are given by:

$$X^{rs} = 2v_sn(rs)\cos\theta | X^\theta = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{42}$$

$$X_{rs} = 2v_sn(rs)\cos\theta | X_\theta = -rs^2v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{43}$$

Note that in the case of variable velocity vs a new set of contravariant and covariant components X^t and X_t appears both in the Natario warp drive equation and in the Natario vector because in this case as the velocity vs changes its value as times goes by due to a constant acceleration a this affects the whole spacetime geometry.The equation for a variable velocity vs due to a constant acceleration a is given by $vs = 2n(rs)at$.

Note also that the remaining contravariant and covariant components of both the Natario warp drive equations and the Natario vectors X^{rs},X_{rs},X^θ and X_θ are defined in function of a constant acceleration a in the case of a variable velocity vs and are defined in function of a constant velocity vs in the case of a constant velocity vs.

But some very important things both equations have in common:

- 1)- Both equations satisfies the Natario definition and condition for a warp drive spacetime using the same Natario shape function $n(rs)$ which gives 0 inside the bubble $\frac{1}{2}$ outside the bubble and $0 < n(rs) < \frac{1}{2}$ in the Natario warped region.
- 2)- The same Natario shape function $n(rs)$ appears in the contravariant and covariant components of both Natario vectors.
- 3)- The same Natario shape function $n(rs)$ appears in the definition of the equation of the variable velocity $vs = 2n(rs)at$

Alcubierre used the original $3+1$ ADM formalism in his warp drive(see eq 1 pg 3 in [1])(see Appendix E) and we have reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to derive the original Natario warp drive equation with constant velocity vs:

$$ds^2 = (1 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{44}$$

The negative energy density for the Natario warp drive in the original $3+1$ ADM formalism in the International System of Units SI (see Appendix G) is given by(see pg 5 in [2])

$$\rho = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{45}$$

In the equatorial plane($1+1$ dimensional spacetime with $rs = x - xs$,$y = 0$ and center of the bubble $xs = 0$)(see Appendix D)

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{46}$$

But for the warp drive equation with variable velocity vs

$$ds^2 = (1 - 2X_t + X_tX^t - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{47}$$

We can say nothing about the negative energy density at first sight and we need to compute "all-the-way-round" the Christoffel symbols Riemann and Ricci tensors and the Ricci scalar in order to obtain the Einstein tensor and hence the stress-energy-momentum tensor in a long and tedious process of tensor analysis liable of occurrence of calculation errors.

Or we can use computers with programs like $Maple$ or $Mathematica$ (see pgs $[342(b)]$ or $[369(a)]$ in [11], pgs $[276(b)]$ or $[294(a)]$ in [13],pgs $[454, 457, 560(b)]$ or $[465, 468, 567(a)]$ in [14]).

Appendix C pgs $[551-555(b)]$ or $[559-563(a)]$ in [14] shows how to calculate everything until the Einstein tensor from the basic input of the covariant components of the $3+1$ spacetime metric using $Mathematica$.

But since the Natario shape function $n(rs)$ is the same for both equations it is reasonable to suppose that derivatives of first second(or perhaps higher)order will appear in the negative energy density expression for the Natario warp drive with variable velocity and since the derivatives of first or second order for the Natario shape function possesses extremely low values these values can obliterate large terms for velocities vs or large accelerations a.For a detailed study of the derivatives of first and second order of the Natario shape function $n(rs)$ see pgs 10 to 41 in [18]

6 Conclusion:

In this work we demonstrated the new equation for the warp drive spacetime according to Natario with variable velocity vs and constant acceleration a in the $3+1$ ADM formalism:

$$ds^2 = (1 - 2X_t + X_t X^t - X_{rs} X^{rs} - X_\theta X^\theta) dt^2 + 2(X_{rs} drs + X_\theta d\theta) dt - drs^2 - rs^2 d\theta^2 \quad (48)$$

The original equation of the Natario warp drive spacetime with constant velocity vs that obeys the $3+1$ ADM formalism is this one:

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta) dt^2 + 2(X_{rs} drs + X_\theta d\theta) dt - drs^2 - rs^2 d\theta^2 \quad (49)$$

Note that in the case of variable velocity vs a new set of contravariant and covariant components X^t and X_t appears in the Natario warp drive equation because in this case as the velocity vs changes its value as times goes by due to a constant acceleration a this affects the whole spacetime geometry.

A real and fully functional warp drive must encompasses accelerations or de-accelerations in order to go from 0 to 200 times light speed in the beginning of an interstellar journey and to slow down to 0 again in the end of the interstellar journey.

Both the Alcubierre and Natario original geometries encompasses warp drives of constant velocities so we expanded the Natario vector to encompass time coordinate as a new Canonical Basis for the Hodge Star generating an extended version of the original Natario warp drive equation which of course encompasses accelerations or de-accelerations.

In section 2 we presented two Natario shape functions and while one of them makes the Natario warp drive impossible to be physically achieved due to high negative energy density requirements the other makes the Natario warp drive perfectly possible to be achieved because this shape function have a form that allows low and "affordable" negative energy density requirements.Then the form of the shape functions affects the behavior of the Natario warp drive spacetime specially in the Natario warped region.For a better description about how the second Natario shape function reduces the negative energy density requirements in the Natario warp drive see [8],[9],[10] and [18].

In section 3 we presented the detailed mathematical structure of the original equation for the Natario warp drive spacetime metric with constant velocity vs in the $3+1$ ADM formalism and we verified that this equation satisfies the Natario requirements for a warp drive spacetime.

In section 4 we presented the detailed mathematical structure of the equation for the Natario warp drive spacetime metric with variable velocity vs and a constant acceleration a in the $3+1$ ADM formalism and we verified that this equation also satisfies the Natario requirements for a warp drive spacetime.

In section 5 we compared both equations for the Natario warp drive and while the original equation with constant speed vs have the spacetime geometry completely known(eq:Christoffel symbols,Riemann and Ricci tensors,Ricci scalar,Einstein tensor,stress-energy-momentum tensor for negative energy densities etc) the same mathematical entities for the equation with variable speed vs and constant acceleration a remains unknown and must be calculated in a "all-the-way-round" hand by hand or can be obtained using computer programs like $Maple$ or $Mathematica$.

Also in section 5 we pointed out the very important fact that the Natario shape function is identical for both equations so the shape function used to lower the negative energy density to "affordable" levels in the original equation will perhaps be valid also in the new one because the derivatives of the shape function possesses extremely low values.

The Natario warp drive spacetime is a very rich environment to study the superluminal features of General Relativity because now we have two spacetime metrics and not only one and the geometry of the new equation in the $3 + 1$ spacetime is still unknown and needs to be cartographed.

Because collisions between the walls of the warp bubble and the hazardous particles of the Interstellar Medium(IM) would certainly occurs in a real superluminal interstellar spaceflight we borrowed the idea of Chris Van Den Broeck proposed some years ago in 1999 in order to increase the degree of protection of the spaceship and the crew members in the Natario warp drive equation for constant speed vs(see pg 46 in [18],pg 3 in [19]).

Our idea was to keep the surface area of the bubble exposed to collisions microscopically small avoiding the collisions with the dangerous IM particles while at the same time expanding the spatial volume inside the bubble to a size larger enough to contains a spaceship inside.

A submicroscopic outer radius of the bubble being the only part in contact with our Universe would mean a submicroscopic surface exposed to the collisions against the hazardous IM particles thereby reducing the probabilities of dangerous impacts against large objects (comets asteroids etc) enhancing the protection level of the spaceship and hence the survivability of the crew members.

Any future development for the Natario warp drive must encompass the more than welcome idea of Chris Van Den Broeck and this idea can also be easily implemented in the Natario warp drive with variable velocity.Since the Broeck idea is independent of the Natario geometry wether in constant or variable velocity we did not covered the Broeck idea here because it was already covered in [18] and [19] and in order to discuss the geometry of a Natario warp drive with variable velocity the Broeck idea is not needed here however the Broeck idea must appear in a real Natario warp drive with variable velocity vs concerning realistic superluminal interstellar spaceflights.

But unfortunately although we can discuss mathematically how to reduce the negative energy density requirements to sustain a warp drive we dont know how to generate the shape function that distorts the spacetime geometry creating the warp drive effect.So unfortunately all the discussions about warp drives are still under the domain of the mathematical conjectures.

However we are confident to affirm that the Natario-Broeck warp drive will survive the passage of the Century XXI and will arrive to the Future.The Natario-Broeck warp drive as a valid candidate for faster than light interstellar space travel will arrive to the the Century $XXIV$ on-board the future starships up there in the middle of the stars helping the human race to give his first steps in the exploration of our Galaxy

Live Long And Prosper

7 Appendix A:differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed vs in a R^3 space basis

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [2],eqs 3.135 and 3.137 pg 82(a)(b) in [15],eq 3.72 pg 69(a)(b) in [15]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{50}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{51}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{52}$$

From above we get the following results

$$dr \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{53}$$

$$rd\theta \sim r\sin\theta(d\varphi \wedge dr) \tag{54}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{55}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see pg 8 in [4],eq 3.72 pg 69(a)(b) in [15]):

$$*dr = r^2 \sin\theta(d\theta \wedge d\varphi) \tag{56}$$

$$*rd\theta = r\sin\theta(d\varphi \wedge dr) \tag{57}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{58}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2\sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{59}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{60}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{61}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{62}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi\wedge dr)] \tag{63}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] - [r\sin^2\theta(d\varphi\wedge dr)] \tag{64}$$

We know that the following expression holds true(see pg 9 in [3], eq 3.79 pg 70$(a)(b)$ in [15]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{65}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] + [r\sin^2\theta(dr\wedge d\varphi)] \tag{66}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [2]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{67}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{68}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{69}$$

According to pg 10 in [3],eq 3.90 pg 74$(a)(b)$ in [15] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{70}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \quad (71)$$

Because and according to pg 10 in [3],eqs 3.90 and 3.91 pg 74$(a)(b)$ in [15],tb 3.2 pg 68$(a)(b)$ in [15]:

$$*d(\alpha+\beta) = d\alpha + d\beta \quad (72)$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \quad (73)$$

$$*d(dx) = d(dy) = d(dz) = 0 \quad (74)$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \quad (75)$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \quad (76)$$

And then we derived again the Natario result of pg 5 in [2]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \quad (77)$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [2] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \quad (78)$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \quad (79)$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \quad (80)$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \quad (81)$$

$$2f(r)r^2 sin\theta \cos\theta(d\theta \wedge d\varphi) + 2f(r)r\sin^2\theta(dr \wedge d\varphi) + r^2\sin^2\theta f'(r)(dr \wedge d\varphi) \tag{82}$$

Comparing the above expressions with the Natario definitions of pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2\sin\theta(d\theta \wedge d\varphi) \tag{83}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \sim -r\sin\theta(dr \wedge d\varphi) \tag{84}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{85}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta \wedge d\varphi)] + 2f(r)\ sin\theta[r\sin\theta(dr \wedge d\varphi)] + f'(r)r\sin\theta[r\sin\theta(dr \wedge d\varphi)] \tag{86}$$

$$2f(r)\ cos\theta e_r - 2f(r)\ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{87}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{88}$$

Defining the Natario Vector as in pg 5 in [2] with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{89}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{90}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [2]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{91}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{92}$$

With our pedagogical approaches

$$nX = 2vs(t)f(r)\ cos\theta dr - vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{93}$$

$$nX = -2vs(t)f(r)\ cos\theta dr + vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{94}$$

8 Appendix B:differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed vs or for the first term $vsdx$ from the Natario vector $nX = vsdx + xdvs$(a variable speed) in a R^4 space basis

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [2],eqs 3.135 and 3.137 pg 82$(a)(b)$ in [15],eq 3.74 pg 69$(a)(b)$ in [15]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{95}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r \sin\theta d\varphi) \wedge dr \sim r \sin\theta (dt \wedge d\varphi \wedge dr) \tag{96}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r \sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{97}$$

From above we get the following results

$$dr \sim r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{98}$$

$$rd\theta \sim r \sin\theta (dt \wedge d\varphi \wedge dr) \tag{99}$$

$$r \sin\theta d\varphi \sim r(dt \wedge dr \wedge d\theta) \tag{100}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(pg 8 in [4],eq 3.74 pg 69$(a)(b)$ in [15]):

$$*dr = r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{101}$$

$$*rd\theta = r \sin\theta (dt \wedge d\varphi \wedge dr) \tag{102}$$

$$*r \sin\theta d\varphi = r(dt \wedge dr \wedge d\theta) \tag{103}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta \cos\theta dt \wedge d\theta \wedge d\varphi + r \sin^2\theta dt \wedge dr \wedge d\varphi = d\left(\frac{1}{2} r^2 \sin^2\theta d\varphi\right) \tag{104}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{105}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{106}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{107}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(dt\wedge d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(dt\wedge d\varphi\wedge dr)] \tag{108}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)] - [r\sin^2\theta(dt\wedge d\varphi\wedge dr)] \tag{109}$$

We know that the following expression holds true(see pg 9 in [3],eq 3.79 pg 70$(a)(b)$ in [15])):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{110}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)] + [r\sin^2\theta(dt\wedge dr\wedge d\varphi)] \tag{111}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [2]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{112}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{113}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{114}$$

According to pg 10 in [3],eq 3.90 pg 74$(a)(b)$ in [15] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) \tag{115}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) \quad (116)$$

Because and according to pg 10 in [3],eqs 3.90 and 3.91 pg $74(a)(b)$ in [15],tb 3.3 pg $68(a)(b)$ in [15]::

$$*d(\alpha+\beta) = d\alpha + d\beta \quad (117)$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df\wedge\alpha - f\wedge d\alpha \quad (118)$$

$$*d(dx) = d(dy) = d(dz) = 0 \quad (119)$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = dt\wedge d(\sin^2\theta)\wedge d\varphi - dt\wedge\sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) \quad (120)$$

$$*[d(r^2)d\varphi] = 2rdt\wedge dr\wedge d\varphi - dt\wedge r^2\wedge dd\varphi = 2r(dt\wedge dr\wedge d\varphi) \quad (121)$$

And then we derived again the Natario result of pg 5 in [2]

$$r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + r\sin^2\theta(dt\wedge dr\wedge d\varphi) \quad (122)$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [2] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \quad (123)$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \quad (124)$$

$$f(r)r^2 2sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \quad (125)$$

$$2f(r)r^2 sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dt\wedge dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \quad (126)$$

$$2f(r)r^2 sin\theta \cos\theta(dt \wedge d\theta \wedge d\varphi) + 2f(r)r\sin^2\theta(dt \wedge dr \wedge d\varphi) + r^2\sin^2\theta f'(r)(dt \wedge dr \wedge d\varphi) \tag{127}$$

Comparing the above expressions with the Natario definitions of pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2\sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{128}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \sim -r\sin\theta(dt \wedge dr \wedge d\varphi) \tag{129}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{130}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(dt \wedge d\theta \wedge d\varphi)] + 2f(r)\ sin\theta[r\sin\theta(dt \wedge dr \wedge d\varphi)] + f'(r)r\sin\theta[r\sin\theta(dt \wedge dr \wedge d\varphi)] \tag{131}$$

$$2f(r)\ cos\theta e_r - 2f(r)\ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{132}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{133}$$

Defining the Natario Vector as in pg 5 in [2] with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{134}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{135}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [2]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{136}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{137}$$

With our pedagogical approaches

$$nX = 2vs(t)f(r)\ cos\theta dr - vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{138}$$

$$nX = -2vs(t)f(r)\ cos\theta dr + vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{139}$$

9 Appendix C:differential forms,Hodge star and the mathematical demonstration of the Natario vector $nX = *(vsx)$ for a variable speed vs and a constant acceleration a

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

In the Appendices A and B we gave the mathematical demonstration of the Natario vector nX in the R^3 and R^4 space basis when the velocity vs is constant.Hence the complete expression of the Hodge star that generates the Natario vector nX for a constant velocity vs is given by:

$$nX = *(vsx) = vs * (dx) \tag{140}$$

$$*dx = *d(rcos\theta) = *d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) = *d[f(r)r^2 \sin^2\theta d\varphi] \tag{141}$$

The equation of the Natario vector nX(pg 2 and 5 in [2]) is given by:

$$nX = X^r e_r + X^\theta e_\theta \tag{142}$$

$$nX = X^r dr + X^\theta r d\theta \tag{143}$$

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)] \sin\theta e_\theta \tag{144}$$

$$nX = 2vs(t)f(r)\ cos\theta dr - vs(t)[2f(r) + rf'(r)]r \sin\theta d\theta \tag{145}$$

With the contravariant shift vector components explicitly given by:

$$X^r = 2v_s f(r) \cos\theta \tag{146}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r)) \sin\theta \tag{147}$$

Because due to a constant speed vs the term $x * d(vs) = 0$.Now we must examine what happens when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs * (dx) + x * (dvs) \tag{148}$$

In order to study the term $x * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eqs 10.102 and 10.103 pgs $363(a)(b)$ and $364(a)(b)$ in [15] with the terms $S = u = 1$[7],eq 3.74 pg $69(a)(b)$ in [15],eqs 11.131 and 11.133 with the term $m = 0$[8] pg $417(a)(b)$ in [15].):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{149}$$

$$dt \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{150}$$

The Hodge star operator defined for the coordinate time is given by:(see eq 3.74 pg $69(a)(b)$ in [15]):

$$*dt = r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{151}$$

The valid expression for a variable velocity $vs(t)$ in the Natario warp drive spacetime due to a constant acceleration a must be given by:

$$vs = 2f(r)at \tag{152}$$

Because and considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble where $X = vs(t)$ and $nX = vs(t) * dx + x * d(vs(t)))$ and $f(r) = 0$ for small r(inside the warp bubble where $X = 0$ and $nX = 0$) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pgs 4 and 5 in [2]) and considering also that the Netario warp drive is a ship-frame based coordinates system(a reference frame placed in the center of the warp bubble where the ship resides-or must reside!!) then an observer in the ship inside the bubble sees every point inside the bubble at the rest with respect to him because inside the bubble $vs(t) = 0$ because $f(r) = 0$.

To illustrate the statement pointed above imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The stream varies its velocity with time.The warp bubble in this case is the aquarium and the walls of the aquarium are the walls of the warp bubble-Natario warped region.An observer in the margin of the river would see the aquarium passing by him at a large speed considering a coordinates system(a reference frame) placed in the margin of the river but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Then for the fish any point inside the aquarium is at the rest with respect to him because inside the aquarium $vs = 2f(r)at$ with $f(r) = 0$ and consequently giving a $vs(t) = 0$.Again with respect to the fish the fish "sees" the margin passing by him with a large relative velocity.The margin in this case is the region outside the bubble "seen" by the fish with a variable velocity $vs(t) = v1$ in the time $t1$ and $vs(t) = v2$ in the time $t2$ because outside the bubble the generic expression for a variable velocity vs is given by $vs = 2f(r)at$ and outside the bubble $f(r) = \frac{1}{2}$ giving a generic expression for a variable velocity vs as $vs(t) = at$ and consequently a $v1 = at1$ in the time $t1$ and a $v2 = at2$ in the time $t2$.Then the variable velocity in not only a function of time alone but must consider also the position of the bubble where the measure is being taken wether inside or outside the bubble.So the velocity must also be a function of r.Its total differential is then given by:

$$dvs = 2[atf'(r)dr + f(r)tda + f(r)adt] \tag{153}$$

[7]These terms are needed to deal with the Robertson-Walker equation in Cosmology using differential forms.We dont need these terms here and we can make $S = u = 1$

[8]This term is needed to describe the Dirac equation in the Schwarzschild spacetime we dont need the term here so we can make $m = 1$.Remember also that here we consider geometrized units in which $c = 1$

Applying the Hodge star to the total differential dvs we get:

$$*dvs = 2[atf'(r) * dr + f(r)t * da + f(r)a * dt] \quad (154)$$

But we consider here the acceleration a a constant.Then the term $f(r)tda = 0$ and in consequence $f(r)t * da = 0$.This leaves us with:

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] \quad (155)$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) + f(r)ar^2 \sin\theta(dr \wedge d\theta \wedge d\varphi)] \quad (156)$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)e_r + f(r)ae_t] \quad (157)$$

The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is given by:

$$nX = *(vsx) = vs * (dx) + x * d(vs) \quad (158)$$

The term $*dx$ was obtained in the Appendices A and B as follows:(see pg 5 in [2])

$$*dx = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta \quad (159)$$

The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + x(2[atf'(r)e_r + f(r)ae_t]) \quad (160)$$

But remember that $x = rcos\theta$(see pg 5 in [2]) and this leaves us with:

$$nX = *(vsx) = vs(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \quad (161)$$

But we know that $vs = 2f(r)at$.Hence we get:

$$nX = *(vsx) = 2f(r)at(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \quad (162)$$

Then we can start with a warp bubble initially at the rest using the Natario vector shown above and accelerate the bubble to a desired speed of 200 times faster than light.When we achieve the desired speed we turn off the acceleration and keep the speed constant.The terms due to the acceleration now disappears and we are left again with the Natario vector for constant speeds shown below:

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)] \sin\theta e_\theta \quad (163)$$

Working some algebra with the Natario vector for variable velocities we get:

$$nX = *(vsx) = 2f(r)at(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \tag{164}$$

$$nX = 4f(r)^2 at\ cos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta + 2atf'(r)rcos\theta e_r + 2f(r)rcos\theta ae_t \tag{165}$$

$$nX = 2f(r)rcos\theta ae_t + 4f(r)^2 at\ cos\theta e_r + 2atf'(r)rcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \tag{166}$$

$$nX = 2f(r)rcos\theta ae_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \tag{167}$$

Then the Natario vector for variable velocities defined using contravariant shift vector components is given by the following expressions:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \tag{168}$$

$$nX = X^t dt + X^r dr + X^\theta r d\theta \tag{169}$$

Or being:

$$nX = 2f(r)rcos\theta ae_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \tag{170}$$

$$nX = 2f(r)rcos\theta adt + 2[2f(r)^2 + rf'(r)]atcos\theta dr - 2f(r)at[2f(r) + rf'(r)]r\sin\theta d\theta \tag{171}$$

The contravariant shift vector components are respectively given by the following expressions:

$$X^t = 2f(r)rcos\theta a \tag{172}$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \tag{173}$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)]\sin\theta \tag{174}$$

10 Appendix D:The Natario warp drive negative energy density in Cartezian coordinates

The negative energy density according to Natario is given by(see pg 5 in [2])[9]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \quad (175)$$

In the bottom of pg 4 in [2] Natario defined the x-axis as the polar axis.In the top of page 5 we can see that $x = rs\cos(\theta)$ implying in $\cos(\theta) = \frac{x}{rs}$ and in $\sin(\theta) = \frac{y}{rs}$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \quad (176)$$

Considering motion in the equatorial plane of the Natario warp bubble (x-axis only) then $[y^2 + z^2] = 0$ and $rs^2 = [(x - xs)^2]$ and making $xs = 0$ the center of the bubble as the origin of the coordinate frame for the motion of the Eulerian observer then $rs^2 = x^2$ because in the equatorial plane $y = z = 0$.

Rewriting the Natario negative energy density in cartezian coordinates in the equatorial plane we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \quad (177)$$

[9] $n(rs)$ is the Natario shape function.Equation written in the Geometrized System of Units $c = G = 1$

11 Appendix E:mathematical demonstration of the Natario warp drive equation for a constant speed vs in the original $3+1$ ADM Formalism according to MTW and Alcubierre

General Relativity describes the gravitational field in a fully covariant way using the geometrical line element of a given generic spacetime metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ where do not exists a clear difference between space and time.This generical form of the equations using tensor algebra is useful for differential geometry where we can handle the spacetime metric tensor $g_{\mu\nu}$ in a way that keeps both space and time integrated in the same mathematical entity (the metric tensor) and all the mathematical operations do not distinguish space from time under the context of tensor algebra handling mathematically space and time exactly in the same way.

However there are situations in which we need to recover the difference between space and time as for example the evolution in time of an astrophysical system given its initial conditions.

The $3+1$ ADM formalism allows ourselves to separate from the generic equation $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ of a given spacetime the 3 dimensions of space and the time dimension.(see pg $[64(b)]$ $[79(a)]$ in [12])

Consider a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg $[65(b)]$ $[80(a)]$ in [12].

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg $[65(b)]$ $[80(a)]$ in [12])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg $[66(b)]$ $[81(a)]$ in [12])
(see also fig 21.2 pg $[506(b)]$ $[533(a)]$ in [11] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg $[507(b)]$ $[534(a)]$ in [11])[10]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i, j = 1, 2, 3$ that measures the proper distance between two points inside each hypersurface

- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.

- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

[10] we adopt the Alcubierre notation here

Combining the eqs (21.40),(21.42) and (21.44) pgs $[507, 508(b)]$ $[534, 535(a)]$ in [11] with the eqs (2.2.5) and (2.2.6) pgs $[67(b)]$ $[82(a)]$ in [12] using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{178}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{179}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{180}$$

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{181}$$

But since $dl^2 = \gamma_{ij} dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii} dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{182}$$

$$(dx^i + \beta^i dt)^2 = (dx^i)^2 + 2\beta^i dx^i dt + (\beta^i dt)^2 \tag{183}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\gamma_{ii}\beta^i dx^i dt + \gamma_{ii}(\beta^i dt)^2 \tag{184}$$

$$\beta_i = \gamma_{ii}\beta^i \tag{185}$$

$$\gamma_{ii}(\beta^i dt)^2 = \gamma_{ii}\beta^i\beta^i dt^2 = \beta_i\beta^i dt^2 \tag{186}$$

$$(dx^i)^2 = dx^i dx^i \tag{187}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{188}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{189}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{190}$$

Note that the expression above is exactly the eq (2.2.4) pgs $[67(b)]$ $[82(a)]$ in [12].It also appears as eq 1 pg 3 in [1].

With the original equations of the $3+1$ ADM formalism given below:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{191}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{192}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ii} - \frac{\beta^i\beta^i}{\alpha^2} \end{pmatrix} \tag{193}$$

and suppressing the lapse function making $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{194}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{195}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -1 & \beta^i \\ \beta^i & \gamma^{ii} - \beta^i\beta^i \end{pmatrix} \tag{196}$$

changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-1 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{197}$$

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{198}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{199}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{200}$$

Remember that the equations given above corresponds to the generic warp drive metric given below:

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{201}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{202}$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{203}$$

Comparing all these equations

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{204}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{205}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{206}$$

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{207}$$

With

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{208}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i\beta^i = X_i X^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector.Hence we have:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{209}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{210}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{211}$$

Looking to the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{212}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2]):

$$X^{rs} = 2v_s n(rs) \cos\theta \tag{213}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{214}$$

But remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_{rs} and X_{θ} with $r = rs$ are given by:

$$X_i = \gamma_{ii} X^i \tag{215}$$

$$X_r = \gamma_{rr} X^r = X_{rs} = \gamma_{rsrs} X^{rs} = 2v_s n(rs) \cos\theta = X^r = X^{rs} \tag{216}$$

$$X_{\theta} = \gamma_{\theta\theta} X^{\theta} = rs^2 X^{\theta} = -rs^2 v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{217}$$

The equations of the Natario warp drive in the $3+1$ ADM formalism are given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{218}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{219}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{220}$$

The matrix components 2×2 evaluated separately for rs and θ gives the following results:[11]

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0r} \\ g_{r0} & g_{rr} \end{pmatrix} = \begin{pmatrix} 1 - X_r X^r & X_r \\ X_r & -\gamma_{rr} \end{pmatrix} \tag{221}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0r} \\ g^{r0} & g^{rr} \end{pmatrix} = \begin{pmatrix} 1 & X^r \\ X^r & -\gamma^{rr} + X^r X^r \end{pmatrix} \tag{222}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0\theta} \\ g_{\theta 0} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 - X_\theta X^\theta & X_\theta \\ X_\theta & -\gamma_{\theta\theta} \end{pmatrix} \tag{223}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0\theta} \\ g^{\theta 0} & g^{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & X^\theta \\ X^\theta & -\gamma^{\theta\theta} + X^\theta X^\theta \end{pmatrix} \tag{224}$$

Then the equation of the Natario warp drive spacetime with a constant speed vs in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{225}$$

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drsdt + X_\theta d\theta dt) - drs^2 - rs^2 d\theta^2 \tag{226}$$

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{227}$$

[11] Actually we know that the real matrix is a 3×3 matrix with dimensions t rs and θ.Our 2×2 approach is a simplification

12 Appendix F:mathematical demonstration of the Natario warp drive equation for a variable speed vs and a constant acceleration a in the original $3+1$ ADM Formalism according to MTW and Alcubierre

In the Appendix C we defined a variable bubble velocity vs due to a constant acceleration a as follows:

$$vs = 2n(rs)at \tag{228}$$

And we obtained the Natario vector nX for a Natario warp drive with variable velocities defined as follows:

$$nX = vs(2n(rs)\ cos\theta e_r - [2n(rs) + rsn'(rs)]\sin\theta e_\theta) + rscos\theta(2[atn'(rs)e_r + n(rs)ae_t]) \tag{229}$$

$$nX = 2n(rs)at(2n(rs)\ cos\theta e_r - [2n(rs) + rsn'(rs)]\sin\theta e_\theta) + rscos\theta(2[atn'(rs)e_r + n(rs)ae_t]) \tag{230}$$

$$nX = X^t e_t + X^{rs} e_r + X^\theta e_\theta \tag{231}$$

$$nX = X^t dt + X^{rs} drs + X^\theta rsd\theta \tag{232}$$

Remember that $x = rscos\theta$(see pg 5 in [2]). Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]) we can see that the Natario vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx + x * d(vs)$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2]).Working with some algebra we got:

$$nX = 2n(rs)rscos\theta ae_t + 2[2n(rs)^2 + rsn'(rs)]atcos\theta e_r - 2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta e_\theta \tag{233}$$

$$nX = 2n(rs)rscos\theta adt + 2[2n(rs)^2 + rsn'(rs)]atcos\theta drs - 2n(rs)at[2n(rs) + rsn'(rs)]rs\sin\theta d\theta \tag{234}$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by:

$$X^t = 2n(rs)rscos\theta a \tag{235}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{236}$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \tag{237}$$

Consider again a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg [65(b)] [80(a)] in [12].Considering now an accelerating warp drive then the amount of time needed for the evolution of the hypersurface from Σ_2 to Σ_3 occurring in the lapse of time $t3$ is smaller than the amount of time needed for the evolution of the hypersurface from Σ_1 to Σ_2 occurring in the lapse of time $t2$ because due to the constant acceleration the speed of the warp bubble is growing from $t2$ to $t3$ and in the lapse of time $t3$ the warp drive is faster than in the lapse of time $t2$.

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg [65(b)] [80(a)] in [12])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg [66(b)] [81(a)] in [12])
(see also fig 21.2 pg [506(b)] [533(a)] in [11] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg [507(b)] [534(a)] in [11])[12]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij} dx^i dx^j$ with $i, j = 1, 2, 3$ that measures the proper distance between two points inside each hypersurface
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.Note that in a warp drive of constant velocity the elapsed times $t2$ and $t3$ are equal because the velocity do not varies between $t2$ and $t3$. Hence the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} is always the same as time goes by but for an accelerating warp drive the elapsed time $t3$ is smaller than the elapsed time $t2$ so the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity generated by a constant acceleration.
- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

Combining the eqs (21.40),(21.42) and (21.44) pgs [507, 508(b)] [534, 535(a)] in [11]
with the eqs (2.2.5) and (2.2.6) pgs [67(b)] [82(a)] in [12] using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{238}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{239}$$

[12] we adopt the Alcubierre notation here

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{240}$$

Remember that in an accelerating warp drive the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity that makes the warp drive moves faster and faster being this velocity generated by the extra terms in the Natario vector.These extra terms must be inserted inside the spacetime metric in $3+1$ using a mathematical structure similar to the one of the lapse function as follows:

$$\alpha^2 = \gamma_{tt}(1+\beta^t)^2 = \gamma_{tt}(1+2\beta^t+\beta^t\beta^t) = (\gamma_{tt}+2\gamma_{tt}\beta^t+\gamma_{tt}\beta^t\beta^t) \tag{241}$$

$$\beta_t = \gamma_{tt}\beta^t \tag{242}$$

Remember that here we are working with geometrized units in which $c=1$ so $\gamma_{tt}=1$

$$\alpha^2 = (1+2\beta_t+\beta_t\beta^t) \tag{243}$$

The spacetime metric in $3+1$ is then given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\gamma_{tt}(1+\beta^t)^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{244}$$

Since $dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{245}$$

$$ds^2 = -\gamma_{tt}(1+\beta^t)^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{246}$$

From the Appendix E we can write the $3+1$ metric as:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{247}$$

Note that the expression above is exactly the eq (2.2.4) pgs [67(b)] [82(a)] in [12].It also appears as eq 1 pg 3 in [1].Changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-\alpha^2 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{248}$$

$$ds^2 = (\alpha^2 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{249}$$

$$ds^2 = (1+2\beta_t+\beta_t\beta^t - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{250}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{251}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1+2\beta_t+\beta_t\beta^t - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{252}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$ and we modified the equation to insert the terms due to the lapse function α^2.(pg 2 in [2])

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{253}$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{254}$$

Comparing all these equations

$$ds^2 = (\alpha^2 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{255}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{256}$$

$$ds^2 = \alpha^2 dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{257}$$

$$\alpha^2 = \gamma_{tt}(1 + \beta^t)^2 \tag{258}$$

$$\alpha^2 = (1 + 2\beta_t + \beta_t\beta^t) \tag{259}$$

$$ds^2 = \gamma_{tt}(1 + \beta^t)^2 dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{260}$$

$$ds^2 = (1 + 2\beta_t + \beta_t\beta^t - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{261}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 + 2\beta_t + \beta_t\beta^t - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{262}$$

With these

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{263}$$

$$ds^2 = \gamma_{tt}(1 - X^t)^2 dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{264}$$

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^t X^t) = (\gamma_{tt} - 2\gamma_{tt}X^t + \gamma_{tt}X^t X^t) = (1 - 2X_t + X_t X^t) \tag{265}$$

The generic equations for the Natario warp drive that obeys the $3+1$ ADM formalism are given below:

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \qquad (266)$$

$$ds^2 = \gamma_{tt}(1 - X^t)^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \qquad (267)$$

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^t X^t) = (\gamma_{tt} - 2\gamma_{tt}X^t + \gamma_{tt}X^t X^t) = (1 - 2X_t + X_t X^t) \qquad (268)$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i\beta^i = X_i X^i$ with X^i being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector both for the spatial components of the Natario vector.In the same way we can see that $\beta^t = -X^t$,$\beta_t = -X_t$ and $\beta_t\beta^t = X_t X^t$ with X^t being the contravariant form of the Natario shift vector and X_t being the covariant form of the Natario shift vector for the time component of the Natario vector.Hence we have:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \qquad (269)$$

$$ds^2 = (1 - 2X_t + X_t X^t - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \qquad (270)$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \qquad (271)$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - 2X_t + X_t X^t - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \qquad (272)$$

Looking to the equation of the Natario vector nX:

$$nX = X^t e_t + X^{rs} e_r + X^\theta e_\theta \qquad (273)$$

$$nX = X^t dt + X^{rs} drs + X^\theta rs d\theta \qquad (274)$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by:

$$X^t = 2n(rs) rs cos\theta a \qquad (275)$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \qquad (276)$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \qquad (277)$$

But remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Remember also that $\gamma_{tt} = 1$.Then the covariant shift vector components X_t,X_{rs} and X_θ with $r = rs$ are given by:

$$X_t = \gamma_{tt} X^t \tag{278}$$

$$X_i = \gamma_{ii} X^i \tag{279}$$

$$X_t = \gamma_{tt} X^t = 2n(rs) rs cos\theta a \tag{280}$$

$$X_r = \gamma_{rr} X^r = X_{rs} = \gamma_{rsrs} X^{rs} = X^r = X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{281}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = rs^2 X^\theta = X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]rs^2 \sin\theta \tag{282}$$

The equations of the Natario warp drive in the $3+1$ ADM formalism are given by:

$$ds^2 = (1 - 2X_t + X_t X^t - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{283}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - 2X_t + X_t X^t - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{284}$$

Then the equation of the Natario warp drive spacetime for a variable velocity and a constant acceleration in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - 2X_t + X_t X^t - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drsdt + X_\theta d\theta dt) - drs^2 - rs^2 d\theta^2 \tag{285}$$

$$ds^2 = (1 - 2X_t + X_t X^t - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{286}$$

13 Appendix G:Dimensional Reduction from $\frac{c^4}{G}$ to $\frac{c^2}{G}$

The Alcubierre expressions for the Negative Energy Density in Geometrized Units $c = G = 1$ are given by(pg 4 in [2])(pg 8 in [1]):[13]:

$$\rho = -\frac{1}{32\pi} vs^2 \left[f'(rs)\right]^2 [\frac{y^2 + z^2}{rs^2}] \tag{287}$$

$$\rho = -\frac{1}{32\pi} vs^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2 + z^2}{rs^2}] \tag{288}$$

In this system all physical quantities are identified with geometrical entities such as lengths,areas or dimensionless factors.Even time is interpreted as the distance travelled by a pulse of light during that time interval,so even time is given in lengths.Energy,Momentum and Mass also have the dimensions of lengths.We can multiply a mass in kilograms by the conversion factor $\frac{G}{c^2}$ to obtain the mass equivalent in meters.On the other hand we can multiply meters by $\frac{c^2}{G}$ to obtain kilograms.The Energy Density($\frac{Joules}{meters^3}$)in Geometrized Units have a dimension of $\frac{1}{length^2}$ and the conversion factor for Energy Density is $\frac{G}{c^4}$.Again on the other hand by multiplying $\frac{1}{length^2}$ by $\frac{c^4}{G}$ we retrieve again ($\frac{Joules}{meters^3}$). [14].

This is the reason why in Geometrized Units the Einstein Tensor have the same dimension of the Stress Energy Momentum Tensor(in this case the Negative Energy Density)and since the Einstein Tensor is associated to the Curvature of Spacetime both have the dimension of $\frac{1}{length^2}$.

$$G_{00} = 8\pi T_{00} \tag{289}$$

Passing to normal units and computing the Negative Energy Density we multiply the Einstein Tensor (dimension $\frac{1}{length^2}$) by the conversion factor $\frac{c^4}{G}$ in order to retrieve the normal unit for the Negative Energy Density ($\frac{Joules}{meters^3}$).

$$T_{00} = \frac{c^4}{8\pi G} G_{00} \tag{290}$$

Examine now the Alcubierre equations:

$vs = \frac{dxs}{dt}$ is dimensionless since time is also in lengths.$\frac{y^2+z^2}{rs^2}$ is dimensionless since both are given also in lengths. $f(rs)$ is dimensionless but its derivative $\frac{df(rs)}{drs}$ is not because rs is in meters. So the dimensional factor in Geometrized Units for the Alcubierre Energy Density comes from the square of the derivative and is also $\frac{1}{length^2}$.Remember that the speed of the Warp Bubble vs is dimensionless in Geometrized Units and when we multiply directly $\frac{1}{length^2}$ from the Negative Energy Density in Geometrized Units by $\frac{c^4}{G}$ to obtain the Negative Energy Density in normal units $\frac{Joules}{meters^3}$ the first attempt would be to make the following:

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi} vs^2 \left[f'(rs)\right]^2 [\frac{y^2 + z^2}{rs^2}] \tag{291}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi} vs^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2 + z^2}{rs^2}] \tag{292}$$

[13]See Geometrized Units in Wikipedia

[14]See Conversion Factors for Geometrized Units in Wikipedia

But note that in normal units vs is not dimensionless and the equations above do not lead to the correct dimensionality of the Negative Energy Density because the equations above in normal units are being affected by the dimensionality of vs.

In order to make vs dimensionless again,the Negative Energy Density is written as follows:

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{293}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{294}$$

Giving:

$$\rho = -\frac{c^2}{G}\frac{1}{32\pi}vs^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{295}$$

$$\rho = -\frac{c^2}{G}\frac{1}{32\pi}vs^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{296}$$

As already seen.The same results are valid for the Natario Energy Density

Note that from

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{297}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{298}$$

Making $c = G = 1$ we retrieve again

$$\rho = -\frac{1}{32\pi}vs^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{299}$$

$$\rho = -\frac{1}{32\pi}vs^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{300}$$

14 Remarks

References [11],[12],[13], [14] and [15] are standard textbooks used to study General Relativity and these books are available or in paper editions or in electronic editions all in Adobe PDF Acrobat Reader.

We have the electronic editions of all these books

In order to make easy the reference cross-check of pages or equations specially for the readers of the paper version of the books we adopt the following convention:when we refer for example the pages $[507, 508(b)]$ or the pages $[534, 535(a)]$ in [11] the (b) stands for the number of the pages in the paper edition while the (a) stands for the number of the same pages in the electronic edition displayed in the bottom line of the Adobe PDF Acrobat Reader

15 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[15]
- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[16][17]

[15] special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[16] "Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[17] appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

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