Beal's conjecture as sum of two vectors in a polynomial vector space that defines an identity-proof

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Abstract

Beal's equation is identified as polynomial identity and therefore the variable x is considered a countable variable. The central theme to prove Beal's conjecture here is identifying its numerical solution as a particular solution to the general polynomial identity of $\alpha x^l + \beta x^l = \delta x^l$, where α, β, δ, and $l > 2$ are positive integers. Beal's equation can be represented by the addition of two vectors in the vector space of the set of all polynomials in the form $p(x) = a\, x^l$ for $a \in \mathbb{Q}$ as a subspace of the infinite vector space over $\mathbb{Q}$ of all polynomials with basis $1, x, x^2, \ldots$ with the ordinary addition of polynomials and multiplication by a scalar from $\mathbb{Q}$, where l is particular to any solution to Beal's equation. Accordingly, all three monomials of Beal's equation numerically produce terms of single power by following the rules of exponentiation. Here we look for elements in the $\mathbb{Q}$ field where the rational number can be converted to a number in exponential form that successfully combine with the basis-element x^l.

1. Introduction and conclusion

Beal's conjecture states that if $a^x + b^y = c^z$, where a, b, c, x, y and z are positive integers and x, y, $z > 2$, then a, b, and c have a common "prime" factor. The conjecture was made by math enthusiast Daniel Andrew Beal in 1997 [1]. So far it has been a challenge to the public as well as to mathematicians to prove the conjecture and no counterexample has been successfully presented to disprove it.

Here we emphasize that Algebraic theorems involving numerical operations from arithmetic are generalized to cover non-numerical objects such as polynomials. In that regard, we identify Beal's equation $a^x + b^y \equiv c^z$ as a particular numerical solution to the general polynomial identity $\alpha x^l + \beta x^l = \delta x^l$, where α, β, δ, l are positive integers, and x is the indeterminate whose value must combine with the coefficients of each term to produce a single power term following the rules of exponentiation, and $(\alpha + \beta) = \delta$. The LHS of the polynomial identity represents the sum of two vectors in a polynomial vector space. The variable x here is a bound variable since we identified the polynomial equation as identity.

To disprove the claim that Beal's numerical solution is a particular solution to a polynomial identity, a counter example must be presented of which the two terms on the LHS of Beal's

equation cannot be combined by factorizing the GCF to yield the RHS. In this sense, the LHS of Beal's equation can be treated as the sum of two single same-variable monomials of the same degree that necessarily must produce the RHS of the equation. Following this claim, to produce a numerical solution to Beal's equation one must combine the two LHS terms algebraically to obtain the one-term expression on the RHS. A polynomial GCD of the two monomials must exist then which allows for the process of combining the two LHS expressions into one by exponential rules.

Suppose a solution to Beal's equation produces $c^z = 3^5$, which can in turn be split to $3^2 \cdot 3^3$ and the coefficient term 3^2 expands to $(1 + 2^3)$ producing the exponential-equation,

$$3^3 + 6^3 = 3^5$$

The idea is to consider c^z as an element in a polynomial vector space over the rational numbers $\mathbb{Q}[x]$, i.e. the numerical value of $c^z = 3^5$ is the rational polynomial function ax^3, where x is allowed to assume values in $\mathbb{Q}$, of indeterminate $x = 3$ and coefficient 3^2. For later comparison of addition of fractions, we present the coefficient in exponential form as well as improper fraction, e.g. $3^2 = \frac{27}{3}$. Therefore a solution to Beal's equation represents selective numbers in $\mathbb{Q}$ that meet the requirement of single power terms. Beal's equation then may be represented by the sum of two same degree monomials representing the addition of two vectors in the vector space of all polynomials $p(x) = ax^l$ over $\mathbb{Q}$, being a subspace to the infinite-dimensional vector space of all polynomials over $\mathbb{Q}$ with basis $1, x, x^2$ ...., where l is positive integer indicating a particular solution to Beal's equation. In other words, we are adding two same degree single-variable monomials with coefficients from $\mathbb{Q}$ that produce a sum of same degree monomial with the condition that the numerical valuation leads to single power terms. In the above example, the monomial-equation is,

$$2^3x^3 + x^3 = 3^2x^3$$

The coefficients in the above equation are particular numbers in the rational field, specifically, 2^3 is the improper fraction $\frac{16}{2}$ and 3^2 is $\frac{27}{3}$. A solution to this numerical identity equation requires the indeterminate to be 3 and the specific subspace to be $p(x) = ax^3$. The restriction introduced by Beal's equation of single power terms requires the proper choice of the subspace as well as the

proper choice of coefficients in the rational numbers. The general equation that describes Beal's equation is,

$$\alpha x^l + \beta x^l = \delta x^l \quad \ldots.(1)$$

In equation (1) we are combining two same degree in one-variable monomial functions by the rules of addition of polynomials and multiplication by a scalar. It represents addition of two vectors in the vector space ax^l as a subspace of the general vector space of all polynomials over $\mathbb{Q}$ with basis $1, x, x^2, x^3$ ... Any of Beal's equations then is a solution in the proper subspace. In the example above, the existence of the common basis element x^3 in the equation is a must since we are adding vectors in the subspace ax^3, and the polynomial variable x is a countable number that must have a value since the polynomial equation is identified as an identity. Any other solution to the equation of $x \neq 3$ satisfies the general polynomial identity but does not comply with Beal's equation of single power terms. We can compare Beal's identity (1) with the identity $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$ that produces Pythagorean triples by which we choose the proper value of the variables x and y to comply with the even and odd numbers on the LHS of the identity to produce the proper triples. Likewise, by choosing the proper value of the variable x in Beal's identity as well as the proper coefficients that produce single power terms we can produce Beal's triples of the specified powers.

A similar situation occurs when we add two fractions. In this case we are adding two vectors in the subspace ax_i of the general infinite-dimensional vector space $\mathbb{Q}$ over itself with basis $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ .... While addition of fractions in the physical sense is obvious, similar addition of numbers in exponential form as that of Beal's equation is not so obvious. The following is an example of adding two fractions in the vector space ax_i with the countable variable $x_i = \frac{1}{6}$,

$$\frac{5}{3} + \frac{7}{2} = \left(\frac{5}{3} + \frac{7}{2}\right)\frac{6}{6} = 10\left(\frac{1}{6}\right) + 21\left(\frac{1}{6}\right) = 31\left(\frac{1}{6}\right)$$

In the example above, the identity fractional-equation represents addition of two vectors in the vector space ax_i of numerical values of the sum $10\left(\frac{1}{6}\right) + 21\left(\frac{1}{6}\right)$ of the general identity equation $\alpha x_i + \beta x_i = \delta x_i$. By the use of the rules of addition of fractions and multiplication by a scalar in the vector space ax_i, we take the basis-element as CF and combine the resulting two

terms (coefficients) to obtain a single fractional term of $31\left(\frac{1}{6}\right)$. If the basis-element x_i is cancelled out from both sides of the equation, the equation reduces to the numerical trivial solution of $10 + 21 = 31$ that represents simple addition of elements in the field of the rational numbers. Like the exponential example above (or any solution to Beal's equation), this example of adding fractions, as adding two vectors in a vector space, shows that the existence of the basis-element in the equation is an integrated part of the addition process of elements in the vector space and presents a valid justification of the intrinsic existence of a GCD monomial and the corresponding numerical GCD on the LHS of Beal's equation once the equation is identified as a polynomial identity.

Basically then we can convert Beal's terms to improper fractions and proceed to add two fractions as above which necessarily includes a common factor of the base in exponential form as that of the denominator of the same number expressed as a fraction, e.g. the LHS of Beal's equation $3^3 + 6^3 = 3^5$ can be added as fractions as $\frac{81}{3} + \frac{1296}{6} = \frac{81}{3} + \frac{648}{3} = \frac{729}{3} = \frac{3^6}{3} = 3^5$. Addition of different fractions than those of Beal's equations produces fractions that simply cannot be expressed in single-power exponential forms. To prove Beal's conjecture then it is sufficient to claim that Beal's equation is a polynomial identity; that both of the LHS and the RHS are equal polynomial functions for every x in their domain. In other words, the numerical solution of Beal's equation is a particular solution to the general polynomial identity equation $\alpha x^l + \beta x^l = \delta x^l$ that represents addition of two vectors in the vector space ax^l of polynomials over the field $\mathbb{Q}$ with basis $1, x, x^2, x^3$ ..., where l is positive integer specific to a particular numerical solution of Beal's equation.

2. Proof of Beal's conjecture

Lemma 1 *Let c^z be any number in exponential form such that $z \geq 3$. Then the term is intrinsically a product of two numbers in exponential form.*

Proof. The proof is obvious by the rules of exponentiation.

Corollary 1. Let Beal's equation be $a^x + b^y = c^z$. Then each term of its numerical solution can be represented as a product of two numbers in exponential form.

Proof. The proof follows from Lemma 1 and the restriction of Beal's conjecture that x, y, z are integers ≥ 3.

Proposition 1. *Let **P** be the vector space of all polynomials over* $\mathbb{Q}$ *and power basis* $1, x, x^2, \ldots$, *with the addition operation and scalar multiplication are defined as the usual polynomial operations. Further, let the set of all polynomials of the form* $\mathrm{p}(x) = ax^l$ *for* $a \in \mathbb{Q}$ *and* $l > 2$ *is positive integer. Then for any particular* l *the set of polynomials* $\mathrm{p}(x)$ *is a subspace of* ***P***.

Proof. We check the criteria of $p(x) = ax^l$ for $l = 3$,

a. Contains the zero vector

   For all $a \in \mathbb{Q}$

   Let $a = 0$; $p(x) = 0$ is a vector in the set.

   For any $p(x)$: $p(x) + 0 = p(x)$

b. Closed under addition

   Choose a_1x^3 and a_2x^3

   $a_1x^3 + a_2x^3 = (a_1 + a_2)x^3 \in ax^3$

c. Closed under scalar multiplication

   Choose a_1x^3 and the scalar b

   $ba_1x^3 \in ax^3$

We conclude that polynomials $p(x) = ax^l$, where $l = 3$ are subspace of ***P***. Similarly we can prove that $p(x) = ax^l$ is a subspace for any value of $l > 3$.

Proposition 2. *Let the general polynomial identity equation of* $\alpha x^l + \beta x^l = \delta x^l$ *represent addition of two polynomials in the infinite vector space* $p(x)$ *as in proposition 1 with coefficients in* $\mathbb{Q}$. *Then the solution to the polynomial equation is every integer value of the indeterminate* x.

Proof. The proof is obvious by the rules of addition of polynomials since the polynomial equation is identified as identity and represents the sum of two elements in the subspace $p(x)$ of the general vector space ***P***.

Corollary 2. Let Beal's equation represent the solution to the general polynomial identity equation $\alpha x^l + \beta x^l = \delta x^l$ with the numerical valuation leads to single power terms. Then, there exists a particular solution to Beal's equation where the coefficients of each of the polynomial terms α, β, δ must combine with the numerical value of the basis x^l to produce single power number.

Proof. The proof is straight forward by employing Lemma 1 and corollary 1 and the use of exponential rules since the polynomial identity defines terms in one-variable and that the numerical solution requires combining the coefficients of the terms with x^l.

Corollary 3. Let Beal's equation represent the solution of the general polynomial identity equation of $\alpha x^l + \beta x^l = \delta x^l$ as in Corollary 2. Then the solution to the equation has a common polynomial GCD of x^l and a common base of x with numerical value.

Proof. Since the general polynomial identity that represents Beal's equation must have a common factor of x^l as the basis element in the vector space $p(x)$ as in proposition 1, it follows that the specific solution of Beal's equation must have a common base of the numerical value corresponding to that of the base x.

Argument: By defining Beal's equation as identity we present it as addition of two vectors in a polynomial vector space over $\mathbb{Q}$. Any term in Beal's equation then must be formed by combining the numerical value of the vector's basis element x^l with its coefficient that is derived from the field $\mathbb{Q}$ by the use of exponential rules. It follows that we can place restrictions on the scalar field $\mathbb{Q}$ that only improper fractional elements in the field that may be multiplied by basis-elements in the polynomial vector space to find solutions to Beal's equation of terms expressed in single power form. Therefore, any term in Beal's equation must be of some order of magnitude larger than or equal to its basis element x^l. A side from coprime solutions of the terms in the general equation (1) we can see that the least value of l is 2. This guarantees that for $l > 2$ the equation holds and corollary 3 applies.

Corollary 4. Let Beal's equation represent the solution of the general polynomial identity equation of $\alpha x^l + \beta x^l = \delta x^l$ as in Corollary 2. Then, for $l > 2$ there exists a solution to the equation where the x^l is a GCD polynomial.

Proof. The proof is clear by corollary 3 and the argument above.

End of proof of Beal's conjecture □

In the following section we present examples of numerical solutions of Beal's Diophantine equation. The examples show how we can take the GCD of x^i on the LHS of the equation and combine it with the sum of the coefficients by the power rules to produce the RHS.

3. Examples of Beal's identity solutions

Example 3.1 The equation $70^3 + 105^3 = 35^4$ complies with Beal's conjecture. Factoring the GCD of 35^4 from the LHS we obtain $(3^3 + 2^3)\,35^3$. Simplifying we obtain $35 \cdot 35^3 = 35^4$; the RHS of the equation.

Example 3.2 For the equation $7^6 + 7^7 = 98^3$, taking CD of 7^3 from the LHS we get $(7^3 + 7^4)\,7^3$. The sum of the coefficient terms yields 2744 which can be shaped to 14^3 by taking the third root, which produces the RHS of the equation upon combining the terms by the power rule of the product of two numbers having the same exponent. If we factor out the GCD of 7^6 from the LHS of the equation, the expression becomes $(1 + 7)7^6$ and can further be expressed as $2^3 \cdot 7^6$. Simplifying we get $2^3 \cdot 49^3 = 98^3$, the RHS. This example works with two possible CF because as the GCF of the basis-element x^lcan be shaped to $x^l = x^{2n}$ representing $7^l = x^6$ and $x^n = 7^3$.

Example 3.3 For the equation $34^5 + 51^4 = 85^4$, factoring the GCD 17^4 gives $2^5 \cdot 17 \cdot 17^4 + 3^4 \cdot 17^4 = 5^4 \cdot 17^4 = 85^4$; the RHS.

Example 3.4 The LHS of the equation $760^3 + 456^3 = 152^4$ can be factored to the product of base primes and becomes $5^3 \cdot 2^9.19^3 + 3^3 \cdot 2^9.19^3$. The two terms now can be combined to yield $(3^3 + 5^3)\,2^9.19^3$, and by shaping 2^9 to 8^3 the expression becomes $(3^3 + 5^3)\,8^3.19^3$ with a GCD of 152^3 to yield $152 \cdot 152^3 = 152^4$; the RHS. The common base-factor of all three terms is 152 which have two distinct prime base-units made of 2 and 19.

Example 3.5 Let's consider the equation $27^4 + 162^3 = 9^7$. By factoring the GCD 27^4 we get $(1 + 2^3)27^4$, which becomes $3^2 \cdot 3^{12}$ and produces 3^{14}, which can be shaped to produce 9^7; the RHS of the equation. It is important to make sure that the sum-term on the RHS of the equation

has not been shaped differently before we judge whether the resulting equation is identical to the given one.

Example 3.6 Another example to beware of the end result as deemed different is the equation $33^5 + 66^5 = 1089^3$. The GCD on the LHS of the equation is 33^5. Simplifying we get $(1 + 32)33^5 = 33^6$, which can easily be shaped to 1089^3; the RHS of the equation. The same goes with the equation $8^3 + 8^3 = 4^5$; we get the sum as 2^{10} or 32^2 which can be shaped to 4^5. The last example simply can be simplified by shaping the terms to $2^9 + 2^9 = 2^{10}$, which simplifies as $2 \cdot 2^9 = 2^{10}$. It is obvious here that both terms on the LHS have the same prime base-unit of 2 no matter what shape the terms take.

Example 3.7 By factoring the GCD of 19^3 from the LHS of the equation $19^4 + 38^3 = 57^3$ we obtain $(19 + 8)\ 19^3$. Simplifying we get $27 \cdot 19^3$ which by shaping 27 becomes $3^3 \cdot 19^3$ and yields the RHS of the equation.

Example 3.8 By factoring out the GCD of 80^{12} from the LHS of the equation $80^{12} + 80^{13} = 1536000^4$ we obtain $(1 + 80)\ 80^{12}$. Simplifying we get $81 \cdot 80^{12}$ which becomes $3^4 \cdot 80^{12}$, and by shaping 80^{12} as 512000^4 we get the RHS of the equation.

Example 3.9 By factoring out the GCD of 28^3 from the LHS of the equation $84^3 + 28^3 = 28^4$ we obtain $(27 + 1)\ 28^3$. Simplifying, we get $28 \cdot 28^3$ which becomes the RHS.

Example 3.10 By factoring out the GCD of 1838^3 from the LHS of the equation $1838^3 + 97414^3 = 5514^4$ we obtain $(1 + 148877)\ 1838^3$. By borrowing 1838 factor from the coefficient term and simplifying we get $81 \cdot 1838^4$. The 81 can be shaped to 3^4 and the product yields the RHS.

Remark we have successfully evaluated every solution of Beal's equation we have checked by the process of factorizing the GCD on the LHS of the equation and by combining the sum of the resulting coefficients with the GCD we always obtained the RHS verifying that all of the equations were identities. The common monomial of the GCD of x^lon the LHS of the general polynomial identity equation of $\alpha x^l + \beta x^l = \delta x^l$ is then the common basis element in the subspace αx^l from the vector space of all polynomials over the field $\mathbb{Q}$ and basis $1, x, x^2, x^3$ ... The value of the variable x gives a common base of the equation in accordance with Beal's conjecture.

4. References

[1] Daniel Mauldin, R. (1997) A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem. Notices of the American Mathematical Society, **44**, 1436-1439