Sketching ‘trinions’ and ‘heptanions’

Kohji Suzuki

kohjisuzuki@yandex.com

Abstract

Attempting to abstract exterior derivative and Hodge star operator, we discuss two number systems sketchily.

1 Introduction

Trying to abstract exterior derivative \((d)\) and Hodge star operator \((\star)\), we deal with them as if they were mere mathematical symbols. In other words, we intentionally forget the well-known and/or minute roles they play in the field of physics for the moment. We frequently use \(\overset{\text{d}}{\text{d}}\) and \(\overset{\star}{\star}\) -inspired \(\overset{\text{\diamond}}{\text{\diamond}}\) symbol \(\overset{\text{\dag}}{\overset{\text{\dag}}{\text{\dag}}}\). These symbols are usually considered to be noninterchangeable. In the meantime, we come up with two number systems, which we tentatively call ‘trinion \((t_r)\)’ and ‘heptanion \((h_e)\)’.

2 Taking a cursory look at \(t_r\)’s and \(h_e\)’s

2.1 \(t_r\)’s

At the outset, we make some definitions. \(\overset{\text{\dag}}{\text{\dag}}\)

\[\text{Definition 2.1.1. } (d \cdot d = dd =) d^2 = 0 \ [2, \ 3].\]

\[\text{Definition 2.1.2. } (\overset{\text{\dag}}{\text{\dag}} \cdot \overset{\text{\dag}}{\text{\dag}} = \overset{\text{\dag}}{\text{\dag}} =) \overset{\text{\dag}}{\text{\dag}}^2 = \pm 1, \text{ and } (\overset{\text{\dag}}{\text{\dag}} \cdot \overset{\text{\dag}}{\text{\dag}} \cdot \overset{\text{\dag}}{\text{\dag}} = \overset{\text{\dag}}{\text{\dag}} \overset{\text{\dag}}{\text{\dag}} \overset{\text{\dag}}{\text{\dag}} =) \overset{\text{\dag}}{\text{\dag}}^4 = 1.\]

\[\text{Definition 2.1.3. } i = \overset{\text{\dag}}{\text{\dag}} \overset{\text{\dag}}{\text{\dag}} \overset{\text{\dag}}{\text{\dag}} d, \text{ whereas } j = \overset{\text{\dag}}{\text{\dag}} d \overset{\text{\dag}}{\text{\dag}} \overset{\text{\dag}}{\text{\dag}}.\]

t\(_r\)’s are a number system whose basis elements are 1, \(i\), \(j\). In addition,

\[\text{Definition 2.1.4. } t_r \ (\overset{\text{\text{\dag}}}{} \overset{\text{\dag}}{\text{\dag}} t_r) \overset{\text{\dag}}{\text{\dag}} a + bi + cj \ (\overset{\text{\text{\dag}}}{} \overset{\text{\dag}}{\text{\dag}} a - bi - cj), \text{ where } a, b, c \text{ belong to the set}\]

\(^1\text{Protein Science Society of Japan}\)

\(^1\)We were inspired by the relation \(\overset{\text{\text{\dag}}}{} \overset{\text{\text{\dag}}}{} \overset{\text{\text{\dag}}}{} \omega = (-1)^k(a - \overset{\text{\dag}}{\text{\dag}} \overset{\text{\dag}}{\text{\dag}} k) \omega. \text{ See, e.g., } [1], \text{ in which the author employs the symbol } * \text{ instead of } \overset{\text{\dag}}{\text{\dag}}; \text{ however.}\)

\(^2\)See 5.1 for details about the origin of \(\overset{\text{\dag}}{\text{\dag}}.\)

\(^3\)For instance, \(\overset{\text{\dag}}{\text{\dag}} d \text{ is regarded as distinct from } d \overset{\text{\dag}}{\text{\dag}}. \text{ But what if we accepted interchangeability? See 5.2.}\)

\(^4\)In what follows, \(\cdot\) denotes multiplication and is often omitted.
of real numbers (\(\mathbb{R}\)).

The set of \(t_r\)'s is denoted by \(\mathbb{T}_r\). Se \((t_r)\) and Vec \((t_r)\) stand for \(a\) and \(bi+cj\), respectively. And we immediately get the following.

\[
i \cdot i = \mathbb{R} \odot \odot \odot d \odot \odot \odot = \odot \odot \odot d \odot \odot \odot = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \od0
\]

Likewise, \(j \cdot j = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \odot \odot d \odot \odot \odot d = \odot \od0
\]

We thus get the table below.

<table>
<thead>
<tr>
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<th>1</th>
<th>(i)</th>
<th>(j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(i)</td>
<td>(j)</td>
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<td>0</td>
</tr>
<tr>
<td>(j)</td>
<td>(j)</td>
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<td>0</td>
</tr>
</tbody>
</table>

Having managed to get the table above without explicit reference to the so-called physics, we wish to remember it and raise a question.

**Question 2.1.5.** Do \(t_r\)'s have physical implications?\(^5\)

### 2.2 \(h_e\)'s

We forget physics again and consider a basis \(e_0, \ldots, e_6\) corresponding to basis elements \(1, i, \ldots, m, n\), respectively. Using the above-mentioned symbols \(d\) and \(\odot\), we define the following.

**Definition 2.2.1.** \(i \equiv \odot, \ j \equiv d, \ k \equiv \odot d, \ \ell \equiv d \odot, \ m \equiv \odot d \odot, \) and \(n \equiv d \odot d\).

\(h_e\)'s are a number system whose basis elements are \(1, i, j, k, \ell, m, n\). The set of \(h_e\)'s is denoted by \(\mathbb{H}_e\).

**Definition 2.2.2.** ‘\(\star\)-rule’ is as follows: Consider the set \(\{i, j, k, \ell, m, n\}\) and its subset which contains at most two elements. We then make some noninterchangeable products consisting of at most two elements chosen from its complement. Exponentiation of each element is acceptable. Such procedures are indicated by endowing the product coming from the subset we considered with subscript ‘\(\star\)’.

To \(p_{ab}\)’s, or products of \(e_a\) and \(e_b\) \((1 \leq a, b \leq 6)\) which are other than \(0, \pm 1, \pm j, \pm k,\) and \(\pm \ell\), we apply the above ‘\(\star\)-rule’ as needed.\(^6\)

**Example 2.2.3.** We can derive \(\ell^3, m^2n^5, n\), and so on from \(ij\), which comes from the subset

\(^5\)See Def. 2.1.3.
\(^6\)See Def. 2.1.2.
\(^7\)See Def. 2.1.1.
\(^8\)For a somewhat similar system of numbers, see [4], in which \(i^2 = j^2 = -1, \) and \(ij = ji = 0\).
\(^9\)By the way, \(*d = \text{curl}[3].\)
\(^10\)The reader is invited to compare footnote 13 with footnote 15.
Then, the relations below follow, to name a few.

\[ p_{01} = e_0 \cdot e_1 = 1 \cdot i = i, \quad p_{12} = e_1 \cdot e_2 = i \cdot j = \kappa \diamond \cdot d = \diamond d = \kappa k, \]

\[ p_{33} = e_3 \cdot e_3 = k \cdot k = \kappa \diamond d \cdot d = \diamond d \diamond d = \kappa \diamond d \diamond (\text{resp. } d \diamond d) = \kappa \text{ in (resp. } m j). \]

Computing the remainder of \( p_{ab} \)'s (\( 0 \leq a, b \leq 6 \)), we get the following.

<table>
<thead>
<tr>
<th>×</th>
<th>1</th>
<th>i</th>
<th>j</th>
<th>k</th>
<th>ℓ</th>
<th>m</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>i</td>
<td>j</td>
<td>k</td>
<td>ℓ</td>
<td>m</td>
<td>n</td>
</tr>
<tr>
<td>i</td>
<td>i</td>
<td>±1</td>
<td>k</td>
<td>±j</td>
<td>m</td>
<td>±ℓ</td>
<td>k²/mj</td>
</tr>
<tr>
<td>j</td>
<td>j</td>
<td>ℓ</td>
<td>0</td>
<td>n</td>
<td>0</td>
<td>ℓ²/ni</td>
<td>0</td>
</tr>
<tr>
<td>k</td>
<td>k</td>
<td>m</td>
<td>0</td>
<td>in/mj</td>
<td>0</td>
<td>iℓ²</td>
<td>0</td>
</tr>
<tr>
<td>ℓ</td>
<td>ℓ</td>
<td>±j</td>
<td>n</td>
<td>0</td>
<td>jm/ni</td>
<td>0</td>
<td>jk²</td>
</tr>
<tr>
<td>m</td>
<td>m</td>
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<td>in/k²</td>
<td>0</td>
<td>k²i</td>
<td>0</td>
<td>k³</td>
</tr>
<tr>
<td>n</td>
<td>n</td>
<td>jm/ℓ²</td>
<td>0</td>
<td>ℓ²j</td>
<td>0</td>
<td>ℓ³</td>
<td>0</td>
</tr>
</tbody>
</table>

After intermittent oblivion of physics, we wish to raise another question.

**Question 2.2.4.** Do \( h_e \)'s have physical implications?  

### 3 Some (attempted) calculations

#### 3.1 \( t_r \)'s

First, we would like to know whether \( t_r \)'s are commutative under multiplication. Let \( t_r, t_{r'} \subseteq T_r \) be given by

\[
\begin{align*}
t_{r1} &= a_1 + b_1 i + c_1 j, \\
t_{r2} &= a_2 + b_2 i + c_2 j,
\end{align*}
\]

Then,

\[ a_1, a_2, b_1, b_2, c_1, c_2 \subseteq \kappa. \]

---

11Untenable are \( i j i, k l m, \) and so forth, though for example, \( n m n \) can become \( m n n = m n² \), which is found to be derivable from \( i j i \), if we accept interchangeability temporarily.
12See Def. 2.2.1.
13In this simple case, we virtually ignore ‘⋆-rule’, since we only need to reference Def. 2.2.1 to replace \( \diamond d \) by \( k \).
14See Def. 2.2.1.
15Things being a bit complicated in this case, products tantamount to \( k \cdot k = k² = \diamond d \diamond d \) include \( \diamond d \cdot \diamond d, \diamond d \cdot \diamond d, \diamond d \cdot \diamond d, \diamond d \cdot \diamond d, \diamond d \cdot \diamond d, \diamond d \cdot \diamond d, \diamond d \cdot \diamond d, \diamond d \cdot \diamond d, \) and \( \diamond d \cdot \diamond d \), which correspond to \( i \cdot j \cdot i \cdot j, k \cdot i \cdot j, i \cdot ℓ \cdot j, i \cdot j \cdot k, k \cdot k, m \cdot j, \) and \( i \cdot n \), respectively. So we try being amenable to \( k² \), which comes from the subset \( \{ k \} \), to prune away the first five of these.
16See Def. 2.2.1.
17Incidentally, \( *d* = \text{div} [3] \).
\[ t_{r1} \cdot t_{r2} = (a_1 + b_1 i + c_1 j) \cdot (a_2 + b_2 i + c_2 j) \]
\[ = a_1 \cdot (a_2 + b_2 i + c_2 j) + b_1 i \cdot (a_2 + b_2 i + c_2 j) + c_1 j \cdot (a_2 + b_2 i + c_2 j) \]
\[ = a_1 a_2 + a_1 b_2 i + a_1 c_2 j + b_1 a_2 + b_1 b_2 i + b_1 c_2 j + c_1 a_2 + c_1 b_2 i + c_1 c_2 j \]
\[ = a_1 a_2 + a_2 b_1 i + a_1 c_2 j + a_2 b_i + a_2 c_1 j \]
\[ = a_1 a_2 + (a_1 b_2 + a_2 b_1) i + (a_1 c_2 + a_2 c_1) j. \] (1)

and

\[ t_{r2} \cdot t_{r1} = (a_2 + b_2 i + c_2 j) \cdot (a_1 + b_1 i + c_1 j) \]
\[ = a_2 \cdot (a_1 + b_1 i + c_1 j) + b_2 i \cdot (a_1 + b_1 i + c_1 j) + c_2 j \cdot (a_1 + b_1 i + c_1 j) \]
\[ = a_2 a_1 + a_2 b_1 i + a_2 c_1 j + b_2 a_1 + b_2 b_1 i + b_2 c_1 j + c_2 j a_1 + c_2 j b_1 i + c_2 j c_1 j \]
\[ = a_2 a_1 + a_2 b_1 i + a_2 c_1 j + a_2 b_i + b_2 c_1 i + b_2 c_1 j + a_1 c_2 j + b_1 c_2 i + c_1 c_2 j \]
\[ = a_1 a_2 + a_2 b_1 i + a_2 c_1 j + a_1 b_2 i + b_1 b_2 i + b_2 c_1 i + b_2 c_1 j + a_1 c_2 j + b_1 c_2 i + c_1 c_2 j \]
\[ = a_1 a_2 + a_1 b_2 i + a_1 c_1 j + a_2 b_i + a_2 c_2 j \]
\[ = a_1 a_2 + (a_1 b_2 + a_2 b_1) i + (a_1 c_2 + a_2 c_1) j. \] (2)

Since (1) = (2), \( t_r \)'s are commutative under multiplication. Next, what about \(|t_r|^2\)? We recall the square of \( \mathbb{C} \), the modulus of complex number, which equals \( \mathbb{C} \cdot \mathbb{C} = \mathbb{C} \cdot \mathbb{C} \), where \( \mathbb{C} \) (resp. \( \mathbb{C} \)) = \( a + b \mathbb{i} \) (resp. \( a - b \mathbb{i} \)), \( a, b \in \mathbb{R} \), and \( \mathbb{C}^2 = -1 \). In the case of \( t_r \)'s,

\[ t_r \cdot \bar{t}_r = (a + b \mathbb{i} + c \mathbb{j}) \cdot (a - b \mathbb{i} - c \mathbb{j}) = a \cdot (a - b \mathbb{i} - c \mathbb{j}) + b \cdot (a - b \mathbb{i} - c \mathbb{j}) + c \cdot (a - b \mathbb{i} - c \mathbb{j}) \]
\[ = a^2 - abi - acj + bia - bici - bicj + cja - cjob - cjc \]
\[ = a^2 - abi - acj + abj + bici - bcij + acj - bci \]
\[ = a^2 - abi - acj + abj + bici - bcij + acj - bci \]
\[ = a^2. \] (3)

We also compute \( \bar{t}_r \cdot t_r \), though it should equal \( t_r \cdot \bar{t}_r \), since \( t_r \)'s have been shown to be commutative under multiplication. Sure enough,

\[ \bar{t}_r \cdot t_r = (a - b \mathbb{i} - c \mathbb{j}) \cdot (a + b \mathbb{i} + c \mathbb{j}) = a \cdot (a + b \mathbb{i} + c \mathbb{j}) - b \cdot (a + b \mathbb{i} + c \mathbb{j}) - c \cdot (a + b \mathbb{i} + c \mathbb{j}) \]
\[ = a^2 + abi + acj - bia - bici - bcij + cja - cjob - cjc \]
\[ = a^2 + abi + acj - bia - bici - bcij + cja - cjob - cjc \]
\[ = a^2, \] (4)

which amounts to (3). Hence, \(|t_r|^2 = t_r \cdot \bar{t}_r = \bar{t}_r \cdot t_r = a^2 \). What about multiplicative inverse \( \frac{1}{t_r} \), then?

We would like to be so careful again that we compute it in two ways and expect both to coincide.

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18 See Table 1.
19 Ditto.
20 See Def. 2.1.4.
21 See Table 1.
22 See Def. 2.1.4.
23 See Table 1.
We have seen that
\[
\vec{h} = \frac{1}{a+bi+cj} = \frac{1}{a(bi+cj) - (a-bi-cj)} = \frac{a-bi-cj}{(a+bi+cj)(a-bi-cj)} = \frac{a-bi-cj}{a^2} = \frac{\vec{h}}{a^2}.
\] (5)
And
\[
\vec{h} = \frac{1}{a+bi+cj} = \frac{(a-bi-cj) \cdot 1}{(a-bi-cj)(a+bi+cj)} = \frac{a-bi-cj}{t \cdot t_r} = \frac{a-bi-cj}{a^2} = \frac{\vec{h}}{a^2}.
\] (6)
As expected, (5) = (6). Hence, \( \vec{h} = \frac{\vec{h}}{a^2} (a \in \mathbb{R}^2) \).

3.2 \( h_{e'} \)'s

We let two elements \( h_{e1}, h_{e2} \in \mathbb{H}_e \) be given by
\[
\begin{align*}
h_{e1} &= a_1 + b_1i + c_1j + d_1k + e_1\ell + f_1m + g_1n, \\
h_{e2} &= a_2 + b_2i + c_2j + d_2k + e_2\ell + f_2m + g_2n,
\end{align*}
\]
a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2, g_1, g_2 \in \mathbb{R}.

But since there are indefinite elements such as \( \pm 1, \pm j \), and so forth in Table 2, something (rather) subtle might turn out to be mandatory even for computing \( h_{e1} \cdot h_{e2} \) (or \( h_{e2} \cdot h_{e1} \)). So for the moment, we would like to content ourselves with simplification of that table, or Tables 4 and 5 in 5.2.2.

4 Discussion

We have seen that \( t_r \)'s are commutative under multiplication like \( \mathbb{R} \) and that \( |t_r|^2 = a^2 \), which coincides with the square of a real number \( a \), irrespective of whether \( \text{Vec} (t_r) = 0 \). Moreover, if \( \text{Vec} (t_r) = 0 \), we have \( \vec{h} = \frac{1}{a} \), which also coincides with the multiplicative inverse of \( a \in \mathbb{R}^2 \). In these respects, \( t_r \)'s are reminiscent of \( \mathbb{R} \). However, complex numbers, whose set is denoted by \( \mathbb{C} \), are commutative under multiplication too. And computations of \( |t_r|^2 \) were performed in a way analogous to \( \mathbb{H}^2 = \mathbb{R} \cdot \mathbb{R} (\text{resp. } \mathbb{R} \cdot \mathbb{R} ) = (a+b\bar{1}) \cdot (a-b\bar{1}) (\text{resp. } (a+b\bar{1}) \cdot (a+b\bar{1}) ) = a^2 + b^2 \). Here we wonder if \( t_r \)'s can ‘outreach’ \( \mathbb{C} \) from a viewpoint of the square of norm and observe that the following interpretation on how to get \( \mathbb{H}^2 \) is also possible.

Interpretation 4.1. \( \mathbb{H}^2 \) is obtained by extracting real part and imaginary part from \( \mathbb{R} \) or \( \mathbb{Q} \) and computing the sum of their squares. That is, we can obviate in-a-sense-naïve computations like \( \mathbb{H}^2 = (a+b\bar{1}) \cdot (a-b\bar{1}) = a \cdot (a-b\bar{1}) + b\bar{1} \cdot (a-b\bar{1}) = \cdots \), if we wish.

Computing \( |t_r|^2 \) in this vein, we directly get \( |t_r|^2 = a^2 + b^2 + c^2 \mathbb{H} \), which we view as the diminution of \( a^2 + b^2 + c^2 + d^2 \), \( a, b, c, d \in \mathbb{R} \), the square of the norm of a quaternion. Then, we notice that \( i, j \in \mathbb{T}_r \) is to \( ij + ji \) what \( i, j \in \mathbb{H} \), the set of quaternions is to \( ij + ji \), since \( ij + ji = 0 + 0 \)

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\(^{24}\) See (3).
\(^{25}\) See Def. 2.1.4.
\(^{26}\) See (4).
\(^{27}\) See Def. 2.1.4.
\(^{28}\) Correspondingly, we have \( \frac{1}{t_r} = \frac{a-bi-cj}{a^2+bi+cj} = \frac{\vec{h}}{a^2+bi+cj} \). Cf. (5) and (6).
\(^{29}\) See Table 1.
Furthermore, Def. 2.1.4 seems to reflect \( a \pm b i \) and/or \( a \pm b i \pm c j \pm d k \) where double-signs correspond. Taken together, \( t_r \)’s might play a role in bridging a ‘gap’ between \( \mathbb{Q} \) and \( \mathbb{H} \), though as of writing, we have no specific answer to Question 2.1.5 with us.

What about \( h_e \)’s? Provided that we forget \( \ell \)’s in Table 4 to strike out its rightmost two columns and lowermost two rows, we get Table 5, where basis elements are 1, i, j, k, which turn our attention to \( \mathbb{H} \). Likewise, at the time of writing, we have no specific answer to Question 2.2.4. Notwithstanding, we suspect that \( h_e \)’s lie somewhere between \( \mathbb{H} \) and octonions.

Acknowledgment. Conceptually, we are obliged to the authors of this book for providing us with impressive contents. Technically, we would like to thank TikZ developers for their indirect help which enabled us to prepare figures in 5.1 for submission.

References


5 Appendix

5.1 Where does the symbol \( \Diamond \) come from?

We abstract \( \star \star = \pm 1 \) from the relation in footnote 1. Dropping its minus sign and squaring both sides, we obtain \( \star \star \star \star \star = 1 \), the left-hand side of which we intuitively replace by \( \Diamond \cdot \Diamond \cdot \Diamond \cdot \Diamond \). Then, we imagine the equation \( x^4 - 1 = 0 \), which we solve in the realm of \( \mathbb{Q} \) to obtain \( x = \pm 1 \) and \( \pm i \). We plot these solutions on the complex plane as shown in the following.
We join the vertices $1, i, -1,$ and $-i$ in Fig. 1 to prepare the square below for the coming abstraction.

Abstracting the square in Fig. 2 yields the symbol $\Diamond$, which suggests the aforementioned four solutions.

$^{30}$Cf. here.
5.2 What if the symbols $d$ and $\diamond$ were interchangeable?

5.2.1 $t, r$'s

For example, we have $i = 31 \diamond \diamond \diamond \diamond = 32 \diamond \diamond \diamond \diamond = 33 \diamond \diamond \diamond \diamond = 34 \diamond \diamond \diamond \diamond = 35 \diamond \diamond \diamond \diamond = 36 \diamond \diamond \diamond \diamond$. Now that $i = j = d$, we get the following.

Table 3. Slight simplification of Table 1

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<th>$i$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>$i$</td>
</tr>
<tr>
<td>$i$</td>
<td>$i$</td>
<td>0</td>
</tr>
</tbody>
</table>

5.2.2 $h_e$'s

We have, e.g., $k = 39 \diamond d = 40 \diamond \ell$, $m = 41 \diamond d = 42 \diamond d = 43 \diamond \diamond \diamond = 44 \diamond \diamond \diamond = 45 \diamond \diamond \diamond = 46 \diamond \diamond \diamond = 47 \diamond \diamond \diamond = 48 \diamond \diamond \diamond = 49 \diamond \diamond \diamond = 50 \diamond \diamond \diamond$. And we get the following.

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31 See Def. 2.1.3.
32 This holds, because $d$ and $\diamond$ are assumed to be interchangeable in this subsection.
33 See Def. 2.1.2.
34 See Def. 2.1.3.
35 See footnote 32.
36 See Def. 2.1.2.
37 We have struck out the rightmost column and the lowermost row of Table 1.
38 Cf. here.
39 See Def. 2.2.1.
40 See footnote 32.
41 See Def. 2.2.1.
42 Ditto.
43 See footnote 32.
44 See Def. 2.1.2.
45 See Def. 2.2.1.
46 Even if we calculate like $m = \diamond \diamond \diamond = d \diamond \diamond = \cdots$, we can get $\pm j$.
47 See Def. 2.2.1.
48 See footnote 32.
49 See Def. 2.1.1.
50 Even if we calculate like $n = d \diamond d = \diamond \diamond d = \cdots$, we are able to get 0. $n$ can thus function as a ‘null basis element’.
Table 4. Simplification of Table 2

<table>
<thead>
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<th>×</th>
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<th>i</th>
<th>j</th>
<th>k (= ℓ)</th>
<th>± j (= m)</th>
<th>0 (= n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>i</td>
<td>j</td>
<td>k</td>
<td>± j</td>
<td>0</td>
</tr>
<tr>
<td>i</td>
<td>i</td>
<td>± 1</td>
<td>k</td>
<td>± j</td>
<td>± k</td>
<td>0</td>
</tr>
<tr>
<td>j</td>
<td>j</td>
<td>k</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>k (= ℓ)</td>
<td>k</td>
<td>± j</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>± j (= m)</td>
<td>± j</td>
<td>± k</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0 (= n)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The table above can be further simplified as follows:

Table 5. H-like simplification of Table 4

<table>
<thead>
<tr>
<th>×</th>
<th>1</th>
<th>i</th>
<th>j</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>i</td>
<td>j</td>
<td>k</td>
</tr>
<tr>
<td>i</td>
<td>i</td>
<td>± 1</td>
<td>k</td>
<td>± j</td>
</tr>
<tr>
<td>j</td>
<td>j</td>
<td>k</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>k</td>
<td>k</td>
<td>± j</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

51In this table, we tried to decrease the number of letters containing ℓ, m, and n by using identifications such as k = ℓ, ± j = m, etc in order to make Table 5 ensue without much difficulty.

52See arguments in 4.