There is a gravitational plane wave pulse incident on a finite mass where conservation of energy and momentum does not hold

Karl De Paepe

Abstract

We present an example of a gravitational plane wave pulse incident on a finite mass where conservation of energy and momentum does not hold.

1 Gravitational plane wave pulse metric

Define \( u = t - x \) and let the metric \( g_{\mu\nu}(u) \) be \([1]\)

\[
\begin{align*}
g_{00}(u) &= -1 \quad g_{11}(u) = 1 \quad g_{22}(u) = [L(u)]^2 e^{2\beta(u)} \quad g_{33}(u) = [L(u)]^2 e^{-2\beta(u)} \\
g_{01}(u) &= g_{02}(u) = g_{03}(u) = g_{12}(u) = g_{13}(u) = g_{23}(u) = 0
\end{align*}
\]

having \( g_{\mu\nu}(u) = \eta_{\mu\nu} \) for \( u < 0 \) and

\[
\frac{d^2L}{du^2}(u) + \left[ \frac{d\beta}{du}(u) \right]^2 L(u) = 0
\]

This metric will satisfy \( R_{\mu\nu} = 0 \). It is the metric of a gravitational plane wave pulse.

2 Proper Lorentz transformation

Consider a coordinate transformation from \( t, x, y, z \) to \( t', x', y', z' \) coordinates that is a composition of a rotation by \( \theta \) about the \( z \) axis followed by a boost by \( 2 \cos \theta/(1 + \cos^2 \theta) \) in the \( x \) direction followed by a rotation by \( \theta + \pi \) about the \( z \) axis. For \( \theta/\pi \) not an integer this is a proper Lorentz transformation such that

\[
\begin{align*}
t &= t'(1 + 2\cot^2 \theta) - 2x' \cot^2 \theta + 2y' \cot \theta \\
x &= 2t' \cot^2 \theta + x'(1 - 2\cot^2 \theta) + 2y' \cot \theta \\
y &= 2t' \cot \theta - 2x' \cot \theta + y' \\
z &= z'
\end{align*}
\]

By (4) and (5) we have \( u = t - x = t' - x' = u' \). For (4)-(7) define the metric \( g'_{\mu\nu}(u) \) by

\[
g'_{\mu\nu}(u) = \frac{\partial x^\alpha}{\partial x'^{\mu}} \frac{\partial x^\beta}{\partial x'^{\nu}} g_{\alpha\beta}(u)
\]

\*k.depaep@utoronto.ca
hence for the metric (1), (2), we get
\begin{align*}
g'_{00}(u) &= -1 - 4[1 - g_{22}(u)] \cot^2 \theta \\
g'_{01}(u) &= 4[1 - g_{22}(u)] \cot^2 \theta \\
g'_{11}(u) &= 1 - 4[1 - g_{22}(u)] \cot^2 \theta \\
g'_{02}(u) &= -g'_{12}(u) = -2[1 - g_{22}(u)] \cot \theta \\
g_{22}(u) &= g_{22}(u) \quad g'_{03}(u) = g'_{13}(u) = g'_{23}(u) = 0 \\
g'_{33}(u) &= g_{33}(u)
\end{align*}
(9)-(14)

Since \( g_{\mu\nu}(u) = \eta_{\mu\nu} \) for \( u < 0 \) we have \( g'_{\mu\nu}(u) = \eta_{\mu\nu} \) for \( u < 0 \). The metric \( g'_{\mu\nu}(u) \) satisfies \( R_{\mu\nu} = 0 \) and \( g'_{\mu\nu}(u) = \eta_{\mu\nu} \) for \( u < 0 \) is then also the metric of a gravitational plane wave pulse.

Let \( t^{\mu\nu} \) be the energy-momentum tensor of the gravitational field. It is determined by the metric and is zero for the Minkowski metric. Let \( t^{\mu\nu}(u) \) be the energy-momentum tensor of the gravitational field determined by the metric \( g_{\mu\nu}(u) \). We have

\begin{align*}
t^{00}(u) &= t^{01}(u) = t^{11}(u) \\
t^{02}(u) &= t^{03}(u) = t^{12}(u) = t^{13}(u) = t^{22}(u) = t^{23}(u) = t^{33}(u) = 0
\end{align*}
(15)

Let \( t^{\mu\nu}(u) \) be the energy-momentum tensor of the gravitational field determined by the metric \( g'_{\mu\nu}(u) \). For the transformation (4)-(7) and by (15) we have

\[ t'^{\mu\nu}(u) = \frac{\partial x'\mu}{\partial x\alpha} \frac{\partial x^\nu}{\partial x^\beta} t^{\alpha\beta}(u) = t^{\mu\nu}(u) \]
(16)

3 Variable G

We will be letting \( G \) be a variable. Let \( G_N \) be Newton’s constant and let \( f(u) \) be a smooth function that is zero for \( u < 0 \) and increasing for \( u > 0 \). Define the metric \( g_{\mu\nu}(G, u) \) by letting \( \beta(u) \) be \( (G/G_N)^2 f(u) \) in (1). Now choose units so that \( G_N = 1 \). We then have by (1)-(3) that there are functions \( W_2(G, u) \) and \( W_3(G, u) \) such that

\begin{align*}
g_{22}(G, u) &= 1 + 2G^2 f(u) + 2G^4 W_2(G, u) \\
g_{33}(G, u) &= 1 - 2G^2 f(u) + 2G^4 W_3(G, u)
\end{align*}
(17)-(18)

In (9)-(14) let

\[ \cot \theta = G^{-1} \]
(19)

and replace \( g_{\mu\nu}(u) \) by \( g_{\mu\nu}(G, u) \) and \( g'_{\mu\nu}(u) \) by \( g'_{\mu\nu}(G, u) \) giving

\begin{align*}
g'_{00}(G, u) &= -1 + 8f(u) + 8G^2 W_2(G, u) \\
g'_{01}(G, u) &= -8f(u) - 8G^2 W_2(G, u) \\
g'_{11}(G, u) &= 1 + 8f(u) + 8G^2 W_2(G, u) \\
g'_{02}(G, u) &= -g'_{12}(G, u) = 4Gf(u) + 4G^3 W_2(G, u) \\
g'_{22}(G, u) &= 1 + 2G^2 f(u) + 2G^4 W_2(G, u) \\
g'_{23}(G, u) &= 1 - 2G^2 f(u) + 2G^4 W_3(G, u) \\
g'_{33}(G, u) &= g'_{13}(G, u) = g'_{23}(G, u) = 0
\end{align*}
(20)-(26)

Define \( \bar{g}_{\mu\nu}(u) \) to be \( g'_{\mu\nu}(0, u) \).

4 Wave and mass

Let the gravitational wave pulse with metric \( g'_{\mu\nu}(G, u) \) having components (20)-(26) be incident on a finite mass \( A \) at rest at the origin. Let

\[ \hat{g}_{\mu\nu}(G, t, x) = \bar{g}_{\mu\nu}(u) + GQ_{\mu\nu}(G, t, x) \]
(27)

\[ \hat{g}_{\mu\nu}(G, t, x) = \bar{g}_{\mu\nu}(u) + GQ_{\mu\nu}(G, t, x) \]
be the metric of this system of wave and $A$ where $GQ_{\mu\nu}(G,t,x)$ is the correction to $\bar{g}_{\mu\nu}(u)$ due to the $G$ dependence of $g'_{\mu\nu}(G,u)$ and the $G$ dependence of the gravitational field of $A$. When $G = 0$ the metric of the wave is $\bar{g}_{\mu\nu}(u)$ and $A$ has no gravitational field.

Let $t^{\mu\nu}(G,u)$ be the energy-momentum tensor of the gravitational field determined by the metric $g_{\mu\nu}(G,u)$ and $\bar{t}^{\mu\nu}(G,u)$ be the energy-momentum tensor of the gravitational field determined by the metric $g'_{\mu\nu}(G,u)$. Letting $G \to 0$ since $(G/G_X)^2 f(u) \to 0$ we have $g_{\mu\nu}(G,u) \to \eta_{\mu\nu}$ as $G \to 0$. Consequently $t^{\mu\nu}(G,u) \to 0$ as $G \to 0$ hence using (16) we have $t^{\mu\nu}(G,u) = \bar{t}^{\mu\nu}(G,u) \to 0$ as $G \to 0$. Since $g'_{\mu\nu}(G,u) \to \bar{g}_{\mu\nu}(u)$ and $\bar{t}^{\mu\nu}(G,u) \to 0$ as $G \to 0$ we can conclude the energy-momentum tensor $\bar{t}^{\mu\nu}(u)$ of the gravitational field determined by the metric $\bar{g}_{\mu\nu}(u)$ is zero.

Let $\hat{T}^{\mu\nu}(G,t,x)$ be the energy-momentum tensor of $A$ and $\bar{t}^{\mu\nu}(G,t,x)$ the energy-momentum tensor of the gravitational field determined by the metric $\bar{g}_{\mu\nu}(G,t,x)$. Since $\bar{g}_{\mu\nu}(G,t,x) \to \bar{g}_{\mu\nu}(u)$ as $G \to 0$ we have $\hat{T}^{\mu\nu}(G,t,x) \to \bar{t}^{\mu\nu}(u) = 0$. Consequently on assuming conservation of energy-momentum we have, as $G \to 0$, that

$$\frac{\partial \hat{T}^{\mu\alpha}}{\partial x^\alpha} = - \frac{\partial \bar{t}^{\mu\alpha}}{\partial x^\alpha} \to 0$$

(28)

5 Contradiction

Let $\hat{\Gamma}_{\alpha\beta}^{\mu}(G,t,x)$ be the affine connection and $\bar{g}(G,t,x)$ the metric determinant both calculated using the metric $\bar{g}_{\mu\nu}(G,t,x)$. Assuming the coordinate free form of conservation of energy and momentum we also have

$$\hat{T}^{\mu\alpha}_{\\phantom{\alpha};\alpha} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \left( \sqrt{-\bar{g}} \hat{T}^{\mu\alpha} \right) + \hat{\Gamma}_{\alpha\beta}^{\mu} \hat{T}^{\alpha\beta} = 0$$

(29)

Define $\bar{T}^{\mu\nu}(t,x)$ to be the limit of $\hat{T}^{\mu\nu}(G,t,x)$ as $G \to 0$. Taking the limit of (29) as $G \to 0$ we have by (28)-(29) and $\bar{g}_{\mu\nu}(G,t,x) \to \bar{g}_{\mu\nu}(u)$ that the first term of (29) goes to zero so from the second term

$$(\bar{T}^{00} - 2\bar{T}^{01} + \bar{T}^{11}) \frac{df}{du} = 0$$

(30)

Now $f(u)$ is an increasing function for $u > 0$ so for $u > 0$

$$\bar{T}^{00} - 2\bar{T}^{01} + \bar{T}^{11} = 0$$

(31)

Before the wave comes in contact with $A$ let $A$ be a perfect fluid at rest with nonzero constant mass density. Position $A$ so that at $t = 0$ and $x = 0$ the wave first comes in contact with $A$. Since all of $A$ is at rest at $t = 0$ we have $\bar{T}^{01}(0) = 0$. Also pressure is zero on the surface of $A$ hence $\bar{T}^{11}(0) = 0$. Consequently we must have by (31) that $\bar{T}^{00}(0) = 0$ but with nonzero mass density $\bar{T}^{00}(0) \neq 0$ which is a contradiction.

6 Conclusion

We presented an example of a gravitational plane wave pulse incident on a finite mass where conservation of energy and momentum does not hold.

References