Multiplicative Versions of Infinitesimal Calculus

What happens when you replace the **summation** of standard integral calculus with **multiplication**?

Compare the **abbreviated** definition of a standard integral

\[
\int f(x)dx = \lim_{\Delta x \to 0} \sum f(x_i) \Delta x
\]

With

\[
\prod f(x) \Delta x = \lim_{\Delta x \to 0} \prod f(x_i) \Delta x
\]
\[
\prod (1 + f(x) \Delta x) = \lim_{\Delta x \to 0} \prod (1 + f(x_i) \Delta x)
\]

Call these later two “integrals” **multigrals of Type I and II.** *(Note: unlike “normal” products, these products are not discrete but continuous over an interval).*

Consider each in turn.

**Multigrals (Type I)**

By standard operations \(\prod f(x) \Delta x = e \int \ln(f(x))dx\)

By not taking limits, a finite product approximation can be obtained.

For example, let \(f(x) = x\) from 0 to 1. Then the Type I multigral of \(x\) from 0 to 1 is:

\[
\prod_{0}^{1} x \Delta x = e \int_{0}^{1} \ln(x)dx = 1/e
\]

This can be approximated by the sequence

\[
\begin{align*}
[(\frac{1}{3}) (\frac{2}{3})] \Delta \frac{1}{2} & = 0.4714...
\\
[(\frac{1}{4}) (\frac{2}{4}) (\frac{3}{4})] \Delta \frac{1}{3} & = 0.4543...
\\
[(\frac{1}{5}) (\frac{2}{5}) (\frac{3}{5}) (\frac{4}{5})] \Delta \frac{1}{4} & = 0.4427...
\\
\vdots \quad \vdots
\\
[(\frac{1}{1000}) (\frac{2}{1000}) ... (\frac{999}{1000})] \Delta \frac{1}{999} & = 0.369123...etc
\end{align*}
\]

Which tends to \(1/e = 0.36788...\) (use Stirling’s Formula to support this).
For $f(x)=\tan(x)$ in radians from 0 to $\pi/2$, 
\[ \int_0^{\pi/2} \tan(x) \, dx = e^0 = 1 \]

And
\[
\begin{align*}
\left[ \tan(\pi/6) \cdot \tan(2\pi/6) \right] \uparrow \left( \frac{1}{2} \right) &= 1 \\
\left[ \tan(\pi/8) \cdot \tan(2\pi/8) \cdot \tan(3\pi/8) \right] \uparrow \left( \frac{1}{3} \right) &= 1 \ldots etc.
\end{align*}
\]

The above approximations of the multigral can be likened to the **mid-point-rule** when approximating standard integrals. Like standard integrals, multiplicative **analogs** of the **Trapezoidal Rule** and **Simpson’s Rule** can be found, like:

**“Simpson’s” Product:**
\[
\prod_{a}^{b} f(x) \uparrow dx \approx
\left\{ \left[ f(a) \cdot f(b) \right] \right\} \left\{ \left[ f(a + \Delta x) \cdot f(a + 3\Delta x) \ldots \right] \uparrow 4 \right\} \uparrow \left( \frac{\Delta x}{3} \right)
\]

Consider the following approximations:

\[ Y = \prod_{1}^{2} x \uparrow dx = e \uparrow \left( \int_{1}^{2} \ln(x) dx = e \uparrow (2 \ln(2) - 2 + 1) = 4 / e = 1.471517765... \right) \]

<table>
<thead>
<tr>
<th>Multiplicative Analog of …..</th>
<th>Mid-point Rule</th>
<th>Trapezoidal Rule</th>
<th>Simpson’s Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta x=1)</td>
<td>1.5</td>
<td>[(1)(2)]^(1/2)</td>
<td>n.a.</td>
</tr>
<tr>
<td>(\Delta x=1/2)</td>
<td>[(1.25)(1.75)]^(1/2) = 1.4790199…</td>
<td>[(1)(2)]^(1/4) * [1.5]^(1/2) = 1.4564753…</td>
<td>[(1)(2)]^(1/2)(1/3) * [1.5]^(4/3)(1/2) = 1.47084…</td>
</tr>
<tr>
<td>(\Delta x=1/3)</td>
<td>[(7/6)(9/6)(11/6)]^(1/3) = 1.474890668…</td>
<td>[(1)(2)]^(1/6) * [4/3][5/3]^(1/3) = 1.46476345…</td>
<td>n.a.</td>
</tr>
</tbody>
</table>

Like standard calculus you can define a multiplicative analog of the derivative (the m-derivative), construct a multiplicative version of the Fundamental Theorem of Calculus, construct a multiplicative analog of Maclaurin’s Series, etc.
The m-derivative for Type I multigrals is:

\[ f'_i (x) = e \uparrow \left( \frac{f'(x)}{f(x)} \right) \]

The Fundamental Theorem is:

\[ \prod_{a}^{b} f'_i (x) \uparrow dx = \prod_{a}^{b} e \uparrow \left( \frac{f'(x)}{f(x)} \right) dx = \frac{f(b)}{f(a)} \]

Compare with the Fundamental Theorem of Standard Calculus:

\[ \int_{a}^{b} f'(x) dx = f(b) - f(a) \]

Small programs can be written to approximate the above results by finite products for those who doubt.

Type I multigrals find application in the area of population dynamics. With stochastic birth- and death-rates, the conventional approach is to use means (ie: expectations). Without migration, mean populations \( E(P) \) remain constant \( \text{iff} \) mean birth-rates \( E(b) = \) mean death-rates \( E(d) \) under the stochastic recursive equation \( P_{n+1} = (1 + b - d) \cdot P_n \).

But, while mathematically correct, this result is misleading.

In certain circumstances, simulations show that mean birth-rates can significantly exceed mean death-rates yet MOST population trials decline, even though the mean population of many trials stays constant. True.

Let \( G(x) = \prod x \uparrow (p(x) dx) \) where \( X = \) the random variable of \((1+b-d)\) and \( p(x) \) is its probability density function. It can be shown that the \text{MODE} of populations \( P_n \) tends to \( \{ G(x) \uparrow n \} \cdot P_0 \) as \( n \to \infty \). In general \( G(x) \) is \( < E(x) = E(1+b-d) \). Thus when \( E(b) = E(d) \), the mode of \( P_n \to 0 \) as \( n \to \infty \) even though \( E(P_n) = P_0 \).

Thus the stochastic recursive equations (where ran# is a random number between 0 and 1)

\[ P_{n+1} = (2.718281828\ldots \cdot \text{ran} \#) \cdot P_n \]
\[ P_{n+1} = (2 \cdot \text{ran} \# + 0.17696) \cdot P_n \]
\[ P_{n+1} = (\text{ran} \# + 0.54421) \cdot P_n \]

are all constant (in the long-term mode) unlike

\[ P_{n+1} = (2 \cdot \text{ran} \#) \cdot P_n \]
\[ P_{n+1} = (\text{ran} \# + 0.5) \cdot P_n \]
which are constant in the long-term mean but tend to zero in the mode. (Try simulating using Excel if you don’t believe).

Now consider…

**Multigrals (Type II)**

Consider \( \prod_{0}^{1} (1 + x^* dx) \) which is the limit of the sequence:

\[
\begin{align*}
(1 + \frac{1}{2} * 1) &= 1.5 \\
(1 + \frac{1}{4} * \frac{1}{2})(1 + \frac{3}{4} * \frac{1}{2}) &= 1.546875 \\
(1 + \frac{1}{6} * \frac{1}{3})(1 + \frac{3}{6} * \frac{1}{3})(1 + \frac{5}{6} * \frac{1}{3}) &= 1.57359671 \\
(1 + \frac{1}{8} * \frac{1}{4})(1 + \frac{3}{8} * \frac{1}{4})(1 + \frac{5}{8} * \frac{1}{4})(1 + \frac{7}{8} * \frac{1}{4}) &= 1.589455605...etc.
\end{align*}
\]

which tends to \( \sqrt{e} = 1.648721271... \)

This is due to the non-standard integral \( \int_{0}^{1} \ln(1 + x^* dx) = 0.5 \)

(which is **not** of the form \( \int f(x) dx \))

and thus

\[
\prod_{0}^{1} (1 + x^* dx) = e \uparrow (\int_{0}^{1} \ln(1 + x^* dx)) = e \uparrow (\int_{0}^{1} x^* dx) = \sqrt{e}
\]

In general,

\[
\prod_{0}^{1} (1 + f(x) dx) = e \uparrow (\int f(x) dx) \quad \text{provided} \quad \int (f(x) dx) \uparrow n = 0 \quad \text{for} \quad n \in \mathbb{N} \geq 2.
\]

Functions \( f(x) \) which fail the later condition appear to be few.

For instance, \( f(x) = 1/x \) fails this test from 0 to 1 as \( \int_{0}^{1} \left(\frac{dx}{x}\right) \uparrow 2 = \pi^2 / 6 \) (look at the limit definition of the integral under equal \( \Delta x \) subintervals to see this).

But for most other functions \( \int (f(x) dx) \uparrow n = 0 \quad \text{for} \quad n \in \mathbb{N} \geq 2 \)

For Type II multigrals, the m-derivative is: \( f_m^0(x) = \frac{f'(x)}{f(x)} \)
And the Fundamental Theorem is:

\[ \int_a^b (1 + f''(x)) \, dx = \int_a^b \left( 1 + \frac{f'(x)}{f(x)} \right) \, dx = \frac{f(b)}{f(a)} \]

Higher order m-derivatives can be also used, like:

\[ \int_a^b (1 + f'''(x)) \, dx = \int_a^b \left( 1 + \frac{f''(x)}{f'(x)} - \frac{f'(x)}{f(x)} \right) \, dx = \frac{f''(b)}{f'(a)} - \frac{f'(b)}{f(a)} \]

And so on. In general, the Fundamental Theorem becomes more complicated for higher order m-derivatives, unlike (say) polynomials with standard calculus. For instance,

\[ f'''(x) = \frac{\left[ \frac{f''(x)}{f'(x)} - \frac{f'(x)}{f(x)} \right]^2 + \left( \frac{f'(x)}{f(x)} \right)^2}{\left[ \frac{f'(x)}{f(x)} \right]^2} \]

And thus

\[ \int_a^b (1 + f'''(x)) \, dx = \int_a^b \left\{ \frac{f''(x)}{f'(x)} - \frac{f'(x)}{f(x)} \right\} \, dx = \frac{f''(b)}{f'(a)} - \frac{f'(b)}{f(a)} \]

For example, let \( f(x) = \ln(x+2) \) then \( f'(x) = 1/(x+2) \), \( f''(x) = -1/(x+2)^2 \), \( f'''(x) = 2/(x+2)^3 \) and \( f(0) = \ln(2) \), \( f(1) = \ln(3) \), etc. Then
\[
\prod_{0}^{1} (1 + f'''(x)dx)
\]
\[
= \prod_{0}^{1} \left( 1 + \left[ \frac{2/(x + 2) \uparrow 3}{1/(x + 2) \ln(x + 2)} + \frac{1/(x + 2) \uparrow 2}{\ln(x + 2)} - \left( \frac{-1/(x + 2) \uparrow 2}{1/(x + 2) \ln(x + 2)} \right) \right] dx \right)
\]
\[
= \prod_{0}^{1} \left\{ 1 - \frac{1}{(x + 2)} \left[ \frac{\ln(x + 2) + \frac{1}{\ln(x + 2)}}{\ln(x + 2) + 1} \right] * dx \right\}
\]
\[
= f''''(1) / f''''(0)
\]
\[
= \left[ \begin{array}{c}
\frac{f''(1)}{f''(0)} - \frac{f''(1)}{f''(0)} \\
\frac{f'''(1)}{f'''(0)} - \frac{f''(0)}{f''(0)}
\end{array} \right]
\]
\[
= (2/3)((1 + 1/\ln(3))/(1 + 1/\ln(2))) = 0.5213474447...
\]

Approximating using N \Delta x subintervals gives:

<table>
<thead>
<tr>
<th>N</th>
<th>Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.5096103</td>
</tr>
<tr>
<td>100</td>
<td>0.520198</td>
</tr>
<tr>
<td>1000</td>
<td>0.5212327</td>
</tr>
</tbody>
</table>

Whacko!

Like standard calculus you can change variables in the standard way:

\[
\prod_{a}^{b} (1 + f(x)dx) \rightarrow \prod_{f(a)}^{f(b)} (1 + u \frac{du}{f'(f^{-1}(u))})
\]

from

\[
\begin{cases}
\text{let } u = f(x) \text{ then } du = f'(x)dx \\
x = a \Rightarrow u = f(a) \\
x = b \Rightarrow u = f(b) \\
dx = du/f'(x) = du/(f'(f^{-1}(u)))
\end{cases}
\]

And thus, for example:

\[
\prod_{a}^{b} (1 + xdx) = \prod_{0}^{1} (1 + ((b - a)x + a) * (b - a)dx)
\]

Product and Quotient Rules for Type I and II multigrals are:
Surprisingly Type II multigrals have the same sort of “Maclaurin’s” Product as Type I. It is

\[ f(x) = f(0)^*e^{\frac{f''(0)}{f(0)}}x + \frac{1}{2!}\left(\frac{f''(0)}{f(0)}\right)^*x^2 + \frac{1}{3!}\left(\frac{f''(0)}{f(0)}\right)^*x^3 + \ldots \]

And the two types of multigral can be related by

\[ \prod f(x)^*dx = \prod (1 + \ln(f(x))dx) \text{ for acceptable } f(x). \]

**Other Types of Multigral**

With type II multigrals, problems arise for functions like \( f(x) = 1/x \) due to the fact that \( \int f(x)dx \uparrow n \neq 0 \text{ for } n \in \mathbb{N} \geq 2 \). But sometimes related multigrals can be evaluated using certain theta functions. For instance,

\[ \prod_0^1 (1 + (\frac{dx}{x}) \uparrow 2) = 5(1 + (\frac{2}{3})^2)(1 + (\frac{2}{5})^2)\ldots = \cosh(\pi) \approx 11.591\ldots \text{ and} \]

\[ \prod_0^1 (1 - (\frac{dx}{x}) \uparrow 4) = -\cosh(\pi) \approx -11.591\ldots \text{ and} \]

\[ \prod_0^1 (1 + (\frac{2dx}{x}) \uparrow 3) = (4 \uparrow 3 + 1) \left[ \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} \right] \frac{\cosh(\pi \sqrt{3})}{\pi} \text{ and the like.} \]

However, these type III multigrals have certain unusual properties like
\[ \prod_{a}^{b} \left( 1 + \left( \frac{dx}{x} \right)^{\uparrow} 2 \right) = \prod_{a}^{b} \left( 1 + \left( \frac{dx}{x} \right)^{\uparrow} 2 \right) \]

\[ \prod_{a}^{b} \left( 1 + \left( \frac{dx}{x} \right)^{\uparrow} 2 \right) = \prod_{a}^{b} \left( 1 + \left( \frac{dx}{x} \right)^{\uparrow} 2 \right) \ldots \text{etc.} \]

So take care when playing around with.

**Type IV Multigrals**

Surprisingly the multigral

\[ \prod_{a}^{b} \left( 1 + x \uparrow \left( \frac{1}{dx} \right) \right) = e^{\uparrow}\left( \frac{\sqrt{e}}{e-1} - \frac{e}{e^{\uparrow}2-1} + \frac{e^{\uparrow}1.5}{e^{\uparrow}3-1} \ldots \right) \approx 2.22 \ldots \text{exists!} \]

This is thanks to the non-standard “standard” integrals of

\[ \int_{0}^{1} x \uparrow \left( \frac{1}{dx} \right) = \frac{\sqrt{e}}{e-1} = 0.959517 \ldots \]

\[ \int_{0}^{1} x \uparrow \left( \frac{k}{dx} \right) = \frac{e^{\uparrow}(k/2)}{(e^{\uparrow}k-1)} \]

These type of multigrals are more restricted (in range) than type I and II, but can still be used to derive certain stochastic limits such as

\[ \text{mod}\left\{ \sum_{i=1}^{n} (\text{ran} \#) \uparrow n \right\} = \frac{\sqrt{e}}{e-1} \approx 0.959517 \ldots \]

\[ \text{mod}\left\{ \prod_{i=1}^{n} (1 + (\text{ran} \#) \uparrow n) \right\} = e^{\uparrow}\left( \frac{\sqrt{e}}{e-1} - \frac{e}{e^{\uparrow}2-1} + \frac{e^{\uparrow}1.5}{e^{\uparrow}3-1} \ldots \right) \approx 2.22 \ldots \]

Where mod is “the mode” and ran# is a random number between 0 and 1.

**Unanswered Questions**

1. How many types of multigrals are there? Do they all have m-derivatives, Fundamental Theorems, analogs of Simpson’s Rule, Maclaurin Series, etc?
2. What do multigrals do in the complex plane?

Answers please. Happy multigrating!

All comments welcome. Please send to: everythingflows@hotmail.com

\[ \prod_{0}^{e} x = \sqrt{2} \]