Betting on a Tossed Fair Coin – some unexpected results

Nothing could be a simpler than betting on a fair coin with fair odds, right? Well, guess again.

There is surprising complexity in such a simple act, as I hope to show below.

1. Not-so-Great Expectations:

Standard practice is to analyse simple games of chance using expectations. But is this adequate?

Take the example of betting a fixed amount (say $10) on a fair coin toss, starting with (say) $100 and always betting on HEADs. The outcome (or payoff) tree for this looks like this:

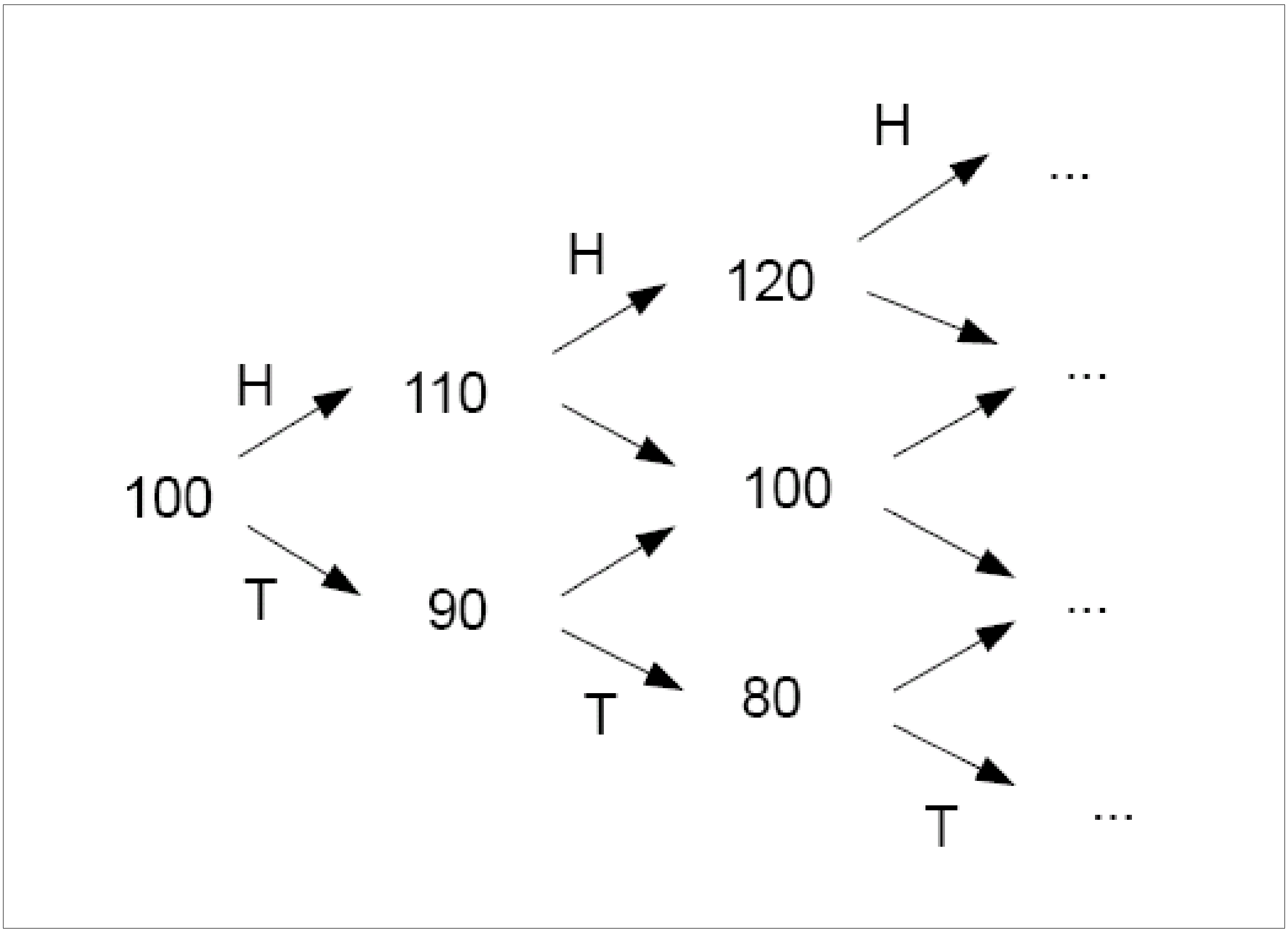
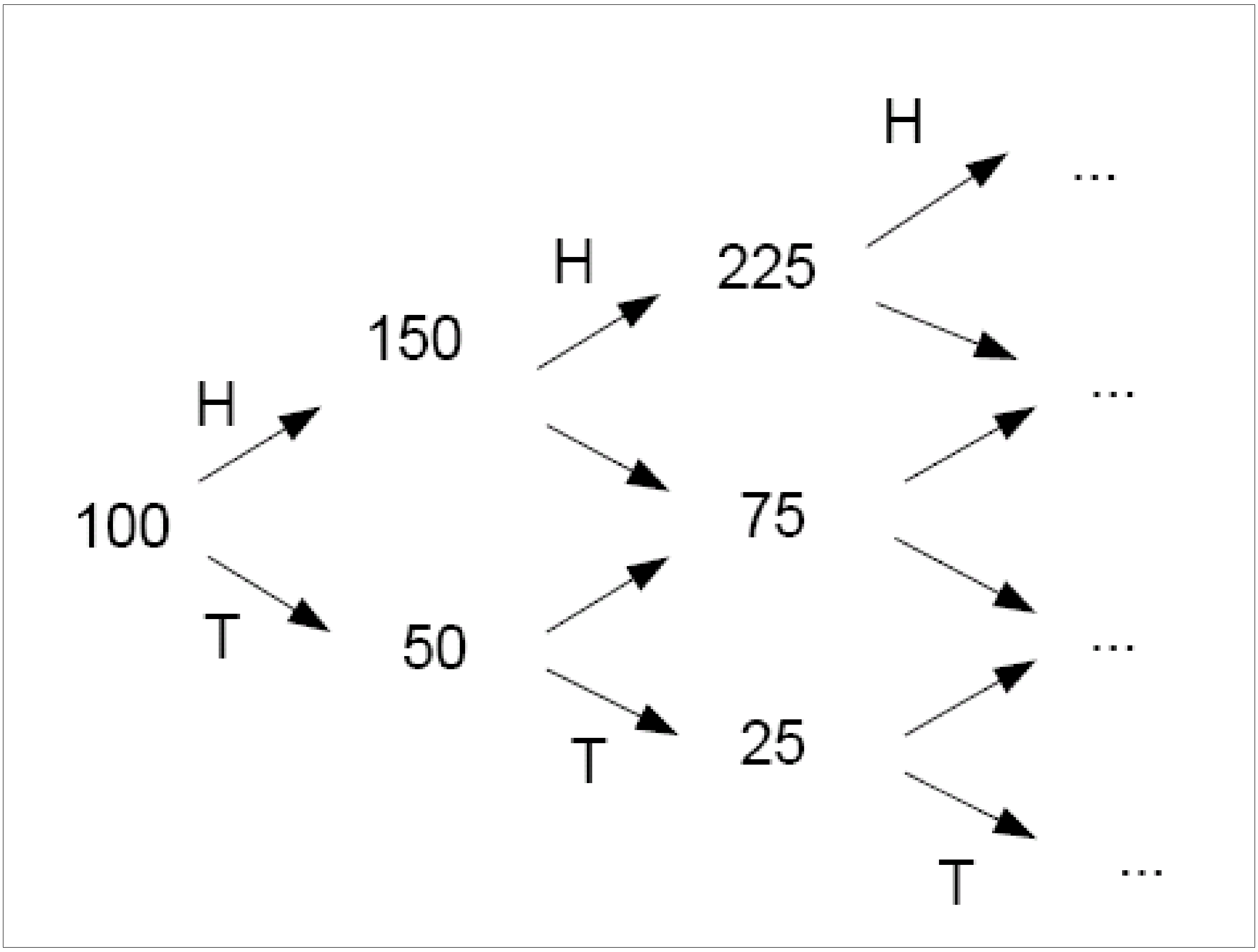


***Figure 1**: Outcome/payoff tree for betting $10 per toss ($ sign to be omitted from all graphs)*

A nice simple "Pascal's Triangle"-like structure as you were probably shown in High School. No surprises.

Now consider the same scenario, only instead of betting a fixed $10 each toss, you bet HALF YOUR CURRENT FUNDS. The outcome tree for this looks like this:



<u>Figure 2</u>: Outcome tree for "bet half of current funds" starting with $100

Okay, we still have the nice "Pascal's Triangle" form … but look at the middle values after 2 tosses (the median). It goes from 100 (the initial value) to 75. Continuing the tree sees it decline ever closer to zero (by 25% after every 2 tosses).

Thus, if you had 2 punters (say **ALICE** and **BOB**) both starting with $100, both betting **HEADS** each toss, with Alice betting a constant $10 while Bob bets **HALF HIS CURRENT FUNDS** then Alice's fund would fluctuate around $100 in a "drunkard's walk" fashion while **BOB GOES BROKE**. True. Calculate a few more branches if you don't believe me.

Counter-intuitive, no?

But it gets worse.

Imagine **CHARLIE** pops up with $100 and bets the **SQUARE ROOT** of his current funds each toss. His outcome tree would look like:

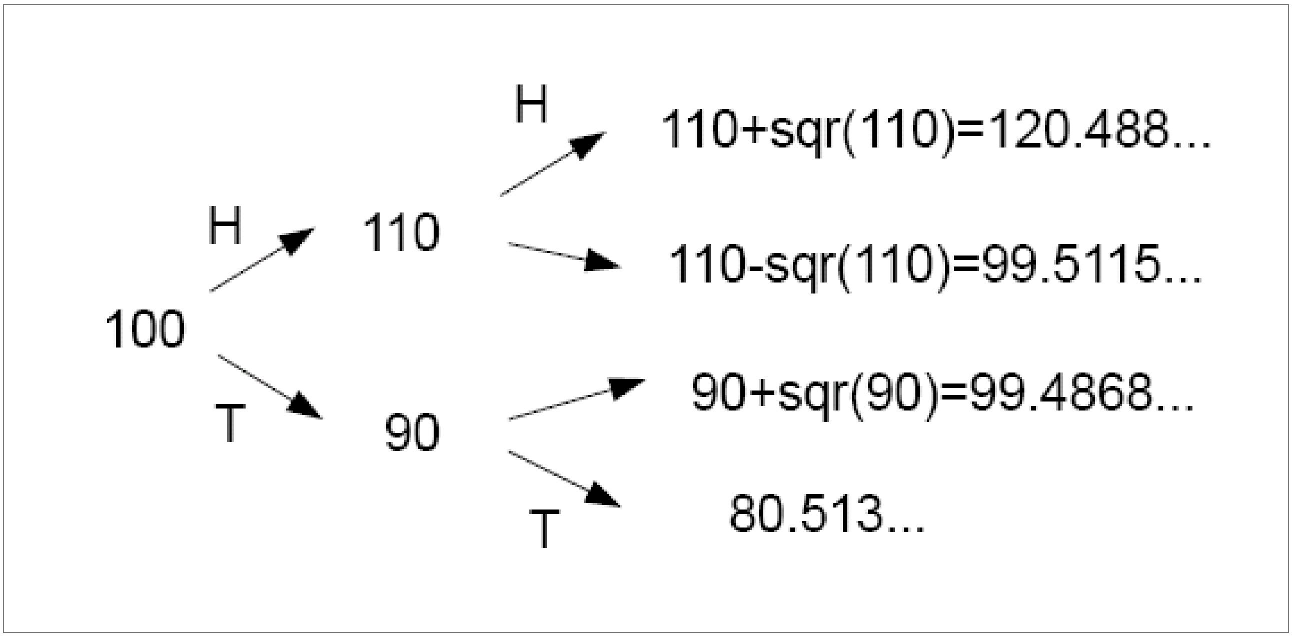


***Figure 3**: Outcome tree for "Bet square root of current funds" starting with $100*

Again the median declines in the long run (from 100 to 99.49...after 2 tosses), but the "Pascal Triangle" form also breaks down giving 4 possible outcomes after 2 tosses instead of 3. Strange, no?

But medians don't just decline. They can rise as well.

Consider **DAVID**. He has $100, always bets **HEADS** (like Alice, Bob and Charlie) but bets "$500 divided by his current funds" [which I'll write as $f(x)=500/x$; $x_0=100$].

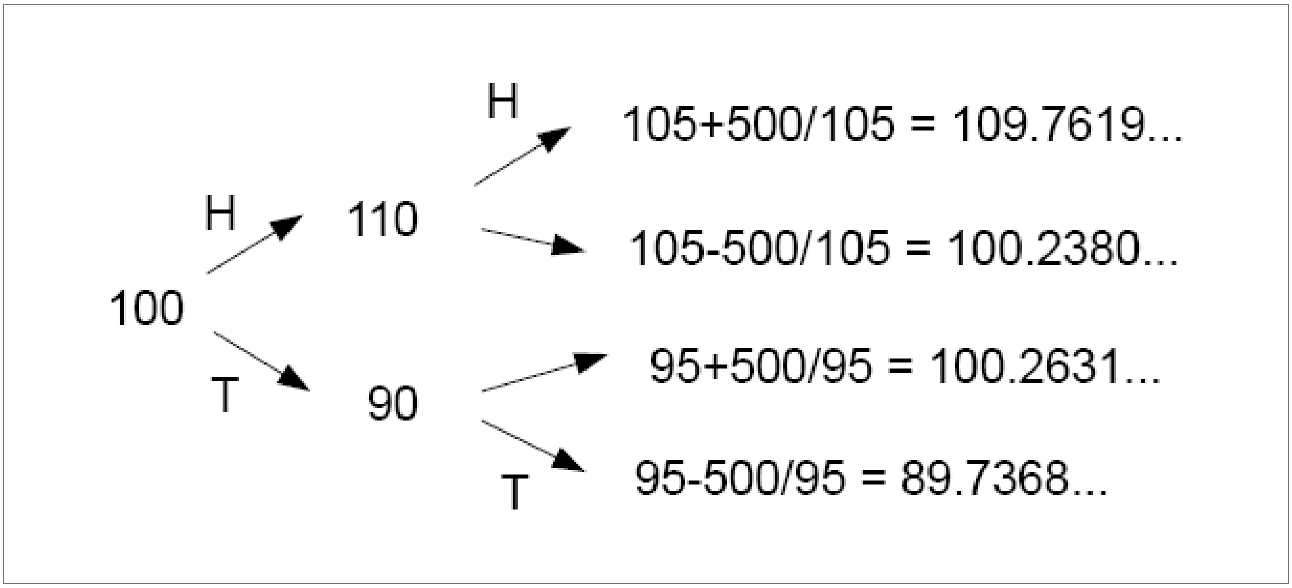


***Figure 4**: Outcome tree for "bet $500/current funds"*

Look at the 2 middle figures after 2 tosses. Both are > 100. This is ***not*** due to rounding error or the like. It's real.

So medians can rise as well as fall for certain betting schemes. Bit different to what you were taught in high school, no?

But wait, there's more.

Consider **ELAINE**. Like the others, she has $100, always bets HEADS but she bets "the square of her current funds/1000" [or f(x)=x*x/1000; x_0=100]. Her outcome tree looks like:

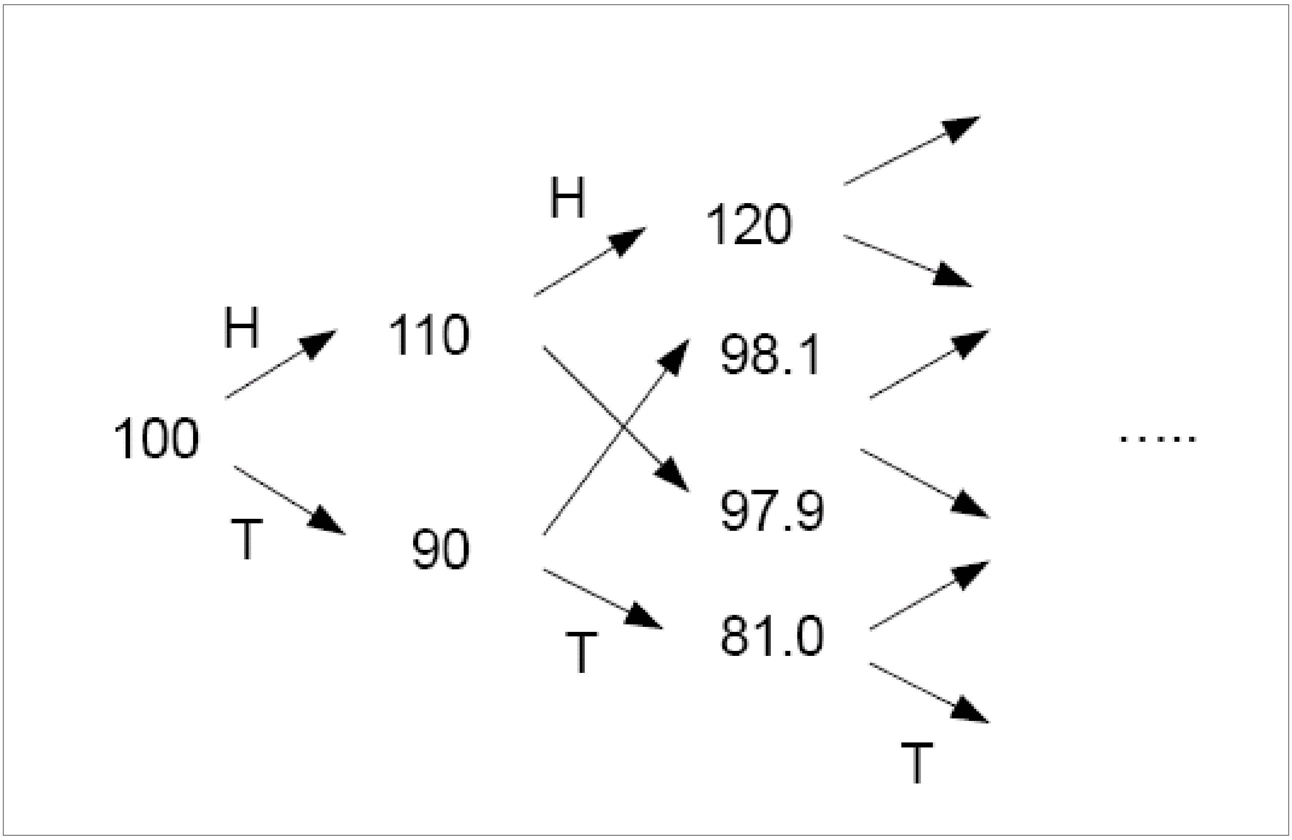


***Figure 4**: Outcome tree for "bet current funds squared/1000"*

Yep, you get crossed branches. So we can't even be sure that outcomes for a number of tosses are ranked from highest down to least. Thus, there's no guarantee that the average of the 2 sequences HTHT... and THTH...(where H=heads and T=tails) will give the median for all betting schemes. Such must be considered an open question for now at least.

So hopefully you are realising that betting on a tossed fair coin isn't as simple as most people (and mathematicians) think. Analysing such problems using expectations alone while being mathematically correct only applies to large aggregates of punters not individuals wishing to analyse their chances.

In general, for betting scheme f(x), the outcome tree goes like:

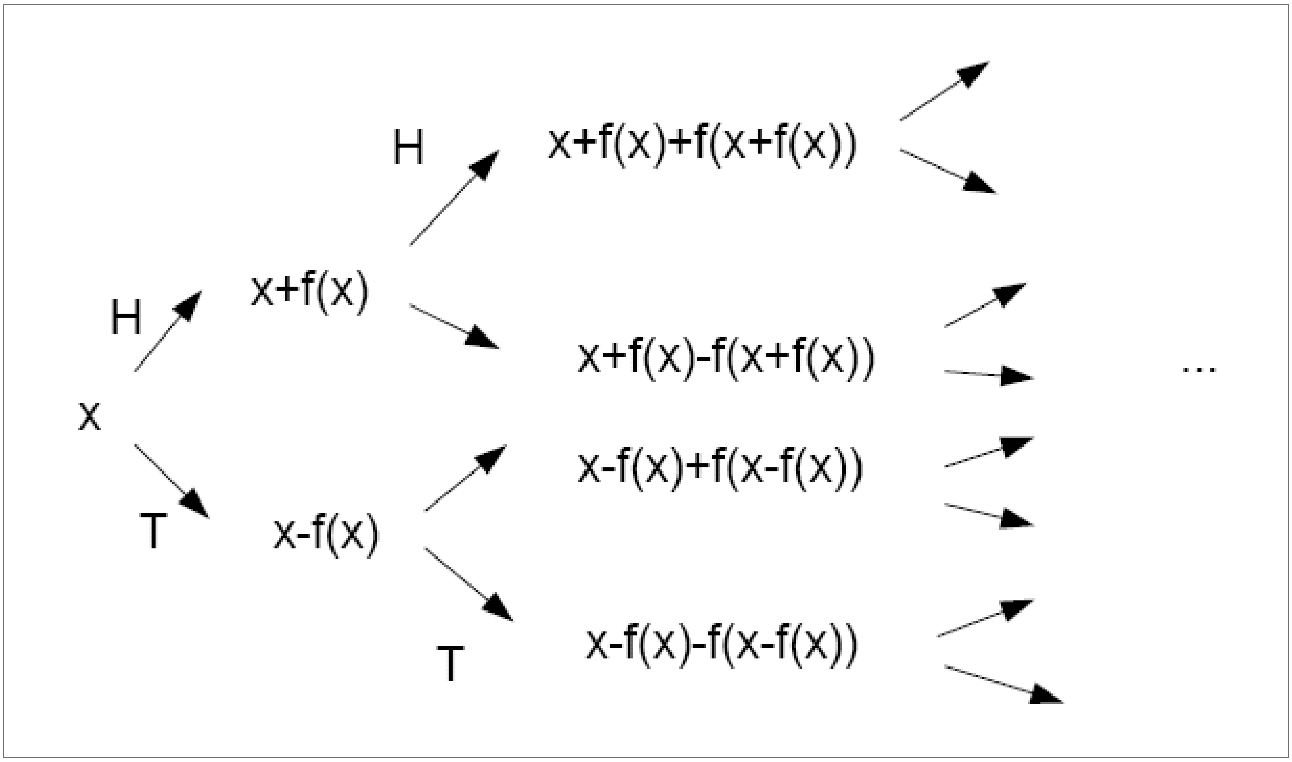


***Figure 6*: Generalised Outcome tree for betting scheme f(x)**

The main question is: given two betting schemes f(x) and g(x) and the amount of initial funds, determine which scheme is better for a given number n of tosses. Sadly, I can't think of a test for this. Instead, the only solution I currently have is to generate the outcome tree from scratch. This can be difficult for even moderate numbers of tosses, especially with spreadsheets. Monte Carlo simulations are another less-than-satisfactory option. But a test would be preferable. Does such exist?

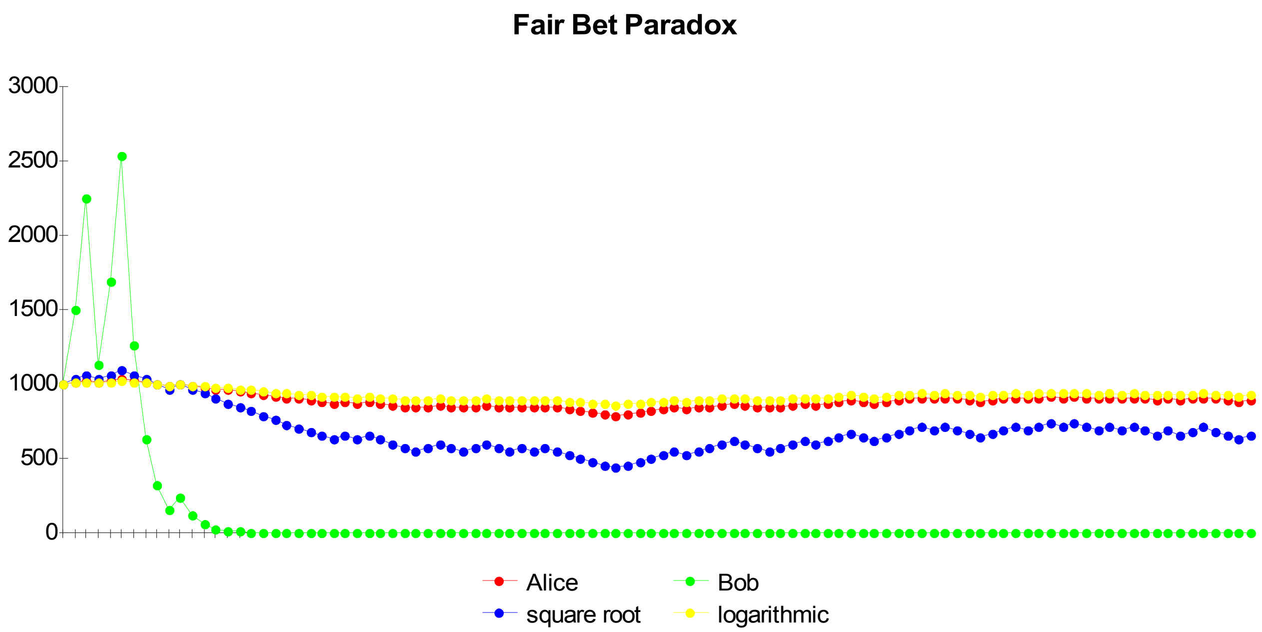


***Figure 7*: Monte Carlo simulations suggest all betting schemes are not equal (see graph above with initial x = 1000).**

For now, I can only categorise certain betting schemes according to the following (limited) criteria:

2. Preliminary Partial Classification of Betting Schemes f(x) on a fair tossed coin (always betting heads)

Type of outcome tree	condition	example
1. HTH...and THT..branches decline	$f(x-f(x))<f(x)<f(x+f(x))$ (includes f(x) such that 1^{st} derivative $f''(x)$ is >0)	$f(x)=sqr(x)$, initial x = 100
2. HTH...and THT...branches increase	$f(x-f(x))>f(x)>f(x+f(x))$ (includes f(x) such that 1^{st} derivative $f'(x)$ is <0)	$f(x)=100/x$, initial x = 20
3. HT and TH branches cross	$f(x)<0.5*[f(x+f(x))+f(x-f(x))]$ (includes f(x) such that 2^{nd} derivative $f''(x)$ is >0)	$f(x)=x^2/1000$, initial x = 100
4. outcome(HT)=outcome(TH)	$\Delta=2*f(x)-[f(x+f(x))+f(x-f(x))]$ $=0$	$f(x)=x/2$, initial x = 100
5. outcome(HT) does not =outcome(TH)	Δ does not =0	See example 1 above

For example, take $f(x)=sqr(x)$, initial x = 100.
Then:

1. $f'(x)$ is > 0 for all $x > 0$, so HTH...and THT... branches decline.
2. $f''(x)$ is > 0 for all $x > 0$, so the HT and TH branches cross.
3. $\Delta=2*f(x)-[f(x+f(x))+f(x-f(x))]$ does not = 0, so outcome(HT) does not = outcome(TH)

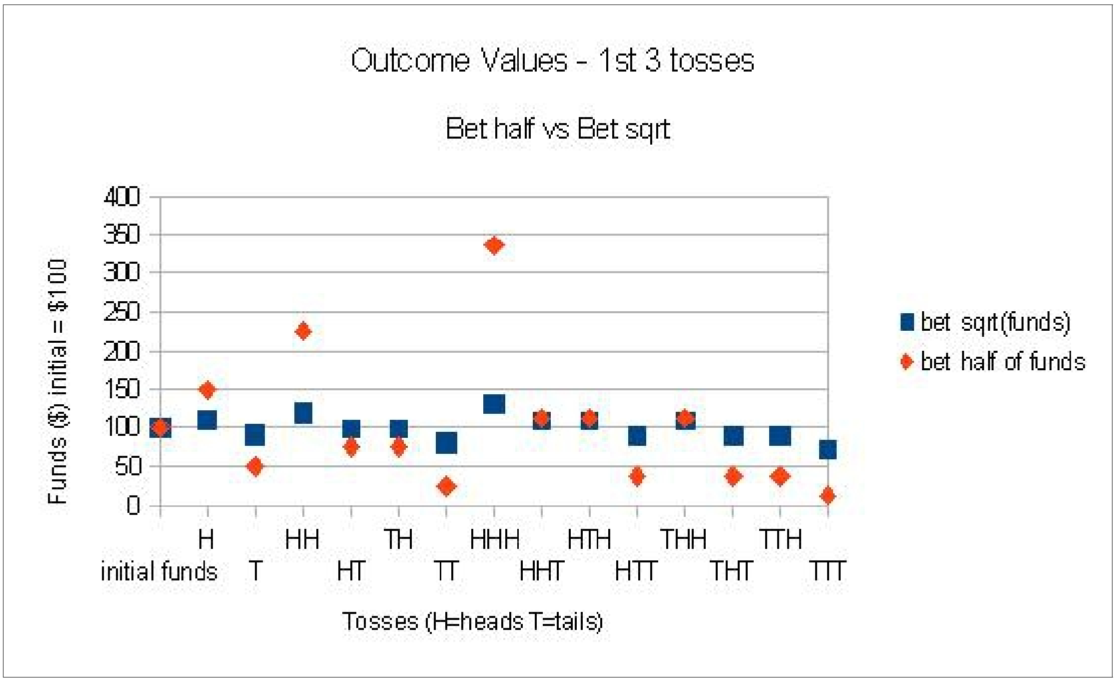
Not much, but something to start with. Hopefully there are other, better tests.

3. What needs doing

Unanswered questions:

1. Is there a way to rank betting schemes for a given number of tosses and initial funds?
2. Is there a way to determine what percentage of time betting scheme A beats betting scheme B given initial funds and number of tosses without starting from scratch? If not in general, how about for various classes of betting scheme?
3. The above two questions when (a) both players bet HEADS, (b) one player bets HEADS and the other TAILS, (c) one player bets HEADS and the other randomly, (d) both players bet randomly.
4. What happens when several players combine and redistribute funds in various ways? (see 'The Dunham Effect' from 'The Fair Bet Paradox')
5. What happens for unfair coins and unfair odds? What about other games of chance? (dice, darts, spinners, roulette, etc)
6. Are there simple programs to generate outcome tree results? What about for spreadsheets?

Notice that even for as few as 3 tosses the two betting schemes "bet half" and "bet square root" show different properties.



For the 14 possible combinations from 3 or less tosses (H, T, HH, TH, …,TTT), "bet half" beats "bet square root" only 6 times out of 14. Does this bias continue? What about for other starting fund values?

		bet sqrt(funds)	bet half of funds
	initial funds	100	100
	H	110	150
1st toss	T	90	50
	HH	120.4880884817	225
	HT	99.5119115183	75
	TH	99.4868329805	75
2nd toss	TT	80.5131670195	25
	HHH	131.4647951136	337.5
	HHT	109.5113818499	112.5
	HTH	109.4874772425	112.5
	HTT	89.5363457941	37.5
	THH	109.4611416272	112.5
	THT	89.5125243338	37.5
	TTH	89.4860799813	37.5
3rd toss	TTT	71.5402540577	12.5

One way to investigate general long-term behaviours might be through HTH...sequences:

$$x \xrightarrow{H} x + f(x) \xrightarrow{T} (x + f(x)) - f(x + f(x)) \xrightarrow{H}$$
$$((x + f(x)) - f(x + f(x))) + f((x + f(x)) - f(x + f(x))) \xrightarrow{T} \ldots$$

And THT...sequences:

$$x \xrightarrow{T} x - f(x) \xrightarrow{H} (x - f(x)) + f(x - f(x)) \xrightarrow{T}$$
$$((x - f(x)) + f(x - f(x))) - f((x - f(x)) + f(x - f(x))) \xrightarrow{H} \ldots$$

There is a lot more to betting on a tossed coin than first meets the eye. The mathematics may be simple but the resulting pattern is not. The problem may be especially suited to high school students to explore, hypothesise, test, prove and generalise given the simplicity of the problem and the maths involved. We should test everything we think we know just in case there is unrecognised pattern lurking about. Remember the Logistic Equation and Chaos!

All feedback welcome.

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