Several Treasures of the Queen of Sciences

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Abstract—**The Guła's Theorem. The proper proof of the Fermat's Last Theorem (FLT). The proof of the Goldbach's Conjecture. The Conclusions. Supplement—the short proof.**

Key Words—**Algebra of sets, Diophantine Equations, Diophantine Inequalities, Exponential Equations, Fermat Equation, Goldbach Conjecture, Greatest Common Divisor, Newton Binomial Formula.**

MSC—**Primary: 11D41, 11P32; Secondary: 11A51, 11D45, 11D61**

I. Introduction

The Guła's theorem [2], [3] and [5] is wider than the Pythagoras's theorem.

In this work we have the proper proof of the famous Fermats Last Theorem.

The Goldbach's Conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes. [1]

II. The Guła's Theorem

Theorem 1 . *For each given* $g \in \{8,12,16,\dots\}$ *or for each given* $g \in \{3,5,7,\dots\}$ *there exist finitely many pairs* (u,v) *of positive integers such that*:

$$g = \left(\frac{g+q^2}{2q}\right)^2 - \left(\frac{g-q^2}{2q}\right)^2 = (u+v)(u-v) = \frac{g}{q}(u-v) = \frac{g}{q}q = g \Longrightarrow g^2 = (u^2-v^2)^2 = (u^2+v^2)^2 - (2uv)^2,$$

where $q \mid g$ *and* $q < \sqrt{g}$ *and* - $q, \frac{g}{q} \in \{2,4,6,\dots\}$ *with even* g *or* $q \in \{1,3,5,\dots\}$ *with odd* g. [2]

III. The Fermat's Last Theorem (FLT)

Theorem 2. *For all* $n \in \{3,4,5,\dots\}$ *and for all* $A,B,C \in \{1,2,3,\dots\}$: $A^n + B^n \neq C^n$.

Proof. Suppose that for some $n \in \{3,4,5,\dots\}$ and for some $A,B,C \in \{1,2,3,\dots\}$: $A^n + B^n = C^n$.

If $A+B \le C$, then $(A^2+B^2 \le C^2 \wedge \dots \wedge A^{n-1} + B^{n-1} \le C^{n-1}) \Longrightarrow A^n + B^n < C^n$,

which is inconsistent with $A^n + B^n = C^n$.

Therefore it must be $(A+B > C \wedge A^2 + B^2 > C^2$ [4]).

Thus – For some $A,B,C,C-A,C-B,v \in \{1,2,3,\dots\}$:

$$A-(C-B) = B-(C-A) = 2v > 0 \Longrightarrow (C-B+2v = A \wedge C-A+2v = B \wedge A+B-2v = C). \qquad (1)$$

At present we can assume for generality of below that A,B and C are coprime. Then only one number out of a hypothetical solutions $[A,B,C]$ is even. Hence we can assume that $A, C-B \in \{1,3,5,\dots\}$.

Let $\{3,5,7,\dots\} - \{(2a+b)b\colon a \in \mathrm{N} \wedge b \in [3,5,7,\dots]\} = \{3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,\dots\} = \mathbb{P}$.

Every even number which is not the power of number 2 has odd prime divisor, hence sufficient that we prove FLT for $n=4$ and for odd prime numbers $n \in \mathbb{P}$. [6]

A. Proof 1 For $n = 4$. If $(A^2)^2 + (B^2)^2 = (C^2)^2$, then

$$[U^2 - V^2 = A^2 \wedge odd\ U-V, u-v \ge 1 \wedge 2UV = B^2 \wedge U^2+V^2 = C^2 \wedge V^2 = (2uv)^2 = U^2 - A^2 = C^2 - U^2 \wedge U = u^2+v^2 \wedge u^2-v^2 = A \wedge \mathbf{gcd}(U,V) = \mathbf{gcd}(u,v) = 1].$$

On the strength of the **Theorem 1** we will have

$$\left[C = \frac{(2uv)^2 + (2v^2)^2}{2 \cdot 2v^2} = u^2+v^2 = U \in \mathbf{0}\right]. \ \square$$

A. 0. Proof 2 For $n = 4$. If $(pq)^4 = C^2 - B^4$, then for some $p,q,C \in \{1,3,5,\dots\}$ and for some $B \in \{2,4,6,\dots\}$ such that p,q,C and B are coprime and $q < p < C$: $(pq)^4 = C^2 - (B^2)^2$.

We assume that the number C is minimal. On the strength of the **Theorem 1** we obtain

$$B^2 = \frac{p^4-q^4}{2} = \frac{p^2+q^2}{2}(p^2-q^2) \Longrightarrow \left(\frac{p^2+q^2}{2} = w^2 \wedge p^2 - q^2 = r^2\right) \Longrightarrow w^2 = \frac{p^2+q^2}{2} = \frac{(u^2+v^2)^2 + (u^2-v^2)^2}{2} = u^4 + v^4 \Longrightarrow w < C,$$

which is inconsistent with minimal C. $\square$

B. Proof For $n \in \mathbb{P}$. Without loss for this proof we can assume that $4 \nmid B, C$. In view of (1) we will have –

For some $n \in \mathbb{P}$ and for some $C, B, C-A \in \{1,2,3,\dots\}$ and for some $C-B, A, v \in \{1,3,5,\dots\}$:

$$\Big[(C-B+2v)^n = (C-B+B)^n - B^n \Longrightarrow (C-B)^{n-2}v + (n-1)(C-B)^{n-3}\,v^2 + \cdots + 2^{n-2}v^{n-1} + \frac{2^{n-1}v^n}{n(C-B)} = \frac{B}{2}\Big[(C-B)^{n-2} + \frac{n-1}{2}(C-B)^{n-3}B + \cdots + B^{n-2}\Big] \wedge\ n \mid v \wedge\ (n \mid B\ \veebar\ n \mid C)\Big] \wedge$$

$$\Big[(C-A+2v)^n = (C-A+A)^n - A^n \Longrightarrow (C-A)^{n-2}2v + \frac{n-1}{2}(C-A)^{n-3}\,(2v)^2 + \cdots + (2v)^{n-1} + \frac{(2v)^n}{n(C-A)} = A\Big[(C-A)^{n-2} + \frac{n-1}{2}(C-A)^{n-3}A + \cdots + A^{n-2}\Big] \wedge\ n \mid v \wedge\ (n \mid A\ \veebar\ n \mid C)\Big] \wedge$$

$$\Bigg[\Big[A^n + B^n = C^n = (A+B-2v)^n = A^n + nA^{n-1}B + \frac{n(n-1)}{2}A^{n-2}B^2 + \cdots + nAB^{n-1} + B^n + n(A+B)^{n-1}\,(-2v) + \frac{n(n-1)}{2}(A+B)^{n-2}(-2v)^2 + \cdots + n(A+B)(-2v)^{n-1} + (-2v)^n \Longrightarrow \Big[0 = AB\frac{\left(A^{n-2} + \frac{n-1}{2}A^{n-3}B + \cdots + B^{n-2}\right)}{A+B} + (A+B)^{n-2}(-2v) + \frac{n-1}{2}(A+B)^{n-3}\,(-2v)^2 + \cdots + (-2v)^{n-1} + \frac{(-2v)^n}{n(A+B)} \wedge\ n \mid v \wedge\ \left(n \mid A\ \veebar\ n \mid B\ \veebar\ (n^{n-1} \mid A+B \wedge\ n \mid C)\right)\Big]\Big]\Bigg].$$

If $n \mid A \equiv 1$, then

$$\big[(n \mid A \veebar n \mid C) \equiv 1 \wedge (n \mid B \veebar n \mid C) \equiv 0 \wedge \left(n \mid A\ \veebar\ n \mid B\ \veebar\ (n^{n-1} \mid A+B\ \wedge\ n \mid C)\right) \equiv 1\big] \in \mathbf{0}.$$

If $n \mid B \equiv 1$, then

$$\big[(n \mid A \veebar n \mid C) \equiv 0 \wedge (n \mid B \veebar n \mid C) \equiv 1 \wedge \left(n \mid A\ \veebar\ n \mid B\ \veebar\ (n^{n-1} \mid A+B\ \wedge\ n \mid C)\right) \equiv 1\big] \in \mathbf{0}.$$

If $n \mid C \equiv 1$, then

$$\big[(n \mid A \veebar n \mid C) \equiv 1 \wedge (n \mid B \veebar n \mid C) \equiv 1 \wedge \left(n \mid A\ \veebar\ n \mid B\ \veebar\ (n^{n-1} \mid A+B\ \wedge\ n \mid C)\right) \equiv 1\big] \in \mathbf{1}.$$

For some $n \in \mathbb{P}$ and for some $e, m, c, h \in \{1,3,5,\dots\}$ such that n, e, m, c and h are coprime:

$$[nemch = v \wedge n \nmid emch \wedge\ (h^n = C-A\ \vee\ 2^n h^n = C-A) \wedge\ c^n = C-B].$$

B.1. Proof For Odd $A, B, C-B$.

For some $n \in \mathbb{P}$ and for some $e, m, c, h \in \{1,3,5,\dots\}$ such that n, e, m, c and h are coprime:

$$[c^n + 2nemch = A \wedge h^n + 2nemch = B \wedge 2^n n^{n-1} m^n = A+B = \ c^n + h^n + 4nemch \wedge\ c^n + B = C] \Longrightarrow [2^n n^{n-1} m^n = \ c^n + h^n + 4nemch \wedge n \mid c+h \wedge\ n^2 \mid c^n + h^n] \Longrightarrow n \mid emch,$$

which is inconsistent with $n \nmid emch$. ♠

B.2. Proof For Even $B, C-A$.

For some $n \in \mathbb{P}$ and for some $e, m, c, h \in \{1,3,5,\dots\}$ such that n, e, m, c and h are coprime:

$$[c^n + 2nemch = A \wedge 2^n h^n + 2nemch = B \wedge\ n^{n-1} m^n = A+B = \ c^n + 2^n h^n + 4nemch \wedge\ c^n + B = C] \Longrightarrow [n^{n-1} m^n = \ c^n + 2^n h^n + 4nemch \wedge\ n \mid c+2h\ \wedge\ n^2 \mid c^n + 2^n h^n\ ] \Longrightarrow n \mid emch,$$

which is inconsistent with $n \nmid emch$. This is the proof.

Remark 1. If $n, A \in \{1,3,5,\dots\}$ and $B \in \{1,2,3,\dots\}$ and $n \mid A+B$ and $\mathbf{gcd}(A,B) = 1$, then

$$\left[\frac{(A+B-B)^n + B^n}{A+B}\ is\ odd\ \wedge\ n^2 \mid A^n + B^n\right],$$

which is obviously. This is the remark.

Theorem 3 . *For each pair* (u, v) *of the relatively prime natural numbers* u *and* v *such that* $u - v$ *is positive and odd there exists exactly one a primitive Pythagorean triple* $(u^2 - v^2, 2uv, u^2 + v^2)$ *and each the primitive Pythagorean triple arises exactly from one pair* (u, v) *of the relatively prime natural numbers* u *and* v *such that* $u - v$ *is positive and odd. Hence – For each equation equations* $(p, q) = (u + v, u - v)$ *of the relatively prime odd natural numbers* p *and* q *such that* $p > q$, *and of the relatively prime natural numbers* u *and* v *such that* $u - v$ *is positive and odd there exists exactly one the primitive Pythagorean triple* $\left(pq, \frac{p^2-q^2}{2}, \frac{p^2+q^2}{2}\right) = (u^2 - v^2, 2uv, u^2 + v^2)$ *and each this primitive Pythagorean triple arises exactly from one equation* $(p, q) = (u + v, u - v)$ *of the relatively prime odd natural numbers* p *and* q *such that* $p > q$, *and of the relatively prime natural numbers* u *and* v *such that* $u - v$ *is positive and odd.* This is the theorem.

IV. PROOF OF THE GOLDBACH'S CONJECTURE

On the strenght of the proof of the Goldbach's Conjecture [2], [3] and of the theorems **1** and **3** we have –

Theorem 4. *For all* $p, q \in \mathbb{P}$ *and for some relatively prime* $u, v \in \{1,2,3, \dots\}$ *such that* $p > q$ *and* $u - v$ is positive and odd: [5]

$$pq = \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2 = u^2 - v^2 = (u+v)(u-v)$$
$$\Rightarrow \left[\left(pq, \frac{p^2-q^2}{2}, \frac{p^2+q^2}{2}\right) = (u^2 - v^2, 2uv, u^2 + v^2) \wedge (p+q)(p-q) = 4uv \wedge p + q = 2u \wedge p - q = 2v \wedge p = u + v \wedge q = u - v \wedge (p + q = 2u = 8,10,12, \dots) \wedge (p - q = 2v = 2,4,6, \dots)\right].$$

Proof. It is easy to verify that

$$4^2 - 1^2 = 5 \cdot 3 \Rightarrow (5 + 3 = 8 \wedge 5 - 3 = 2),$$

$$5^2 - 2^2 = 7 \cdot 3 \Rightarrow (7 + 3 = 10 \wedge 7 - 3 = 4),$$

$$6^2 - 1^2 = 7 \cdot 5 \Rightarrow (7 + 5 = 12 \wedge 7 - 5 = 2),$$

$$7^2 - 4^2 = 11 \cdot 3 \Rightarrow (11 + 3 = 14 \wedge 11 - 3 = 8),$$

$$8^2 - 3^2 = 11 \cdot 5 \Rightarrow (11 + 5 = 16 \wedge 11 - 5 = 6),$$

$$8^2 - 5^2 = 13 \cdot 3 \Rightarrow (13 + 3 = 16 \wedge 13 - 3 = 10),$$

$$9^2 - 2^2 = 11 \cdot 7 \Rightarrow (11 + 7 = 18 \wedge 11 - 7 = 4),$$

$$9^2 - 4^2 = 13 \cdot 5 \Rightarrow (13 + 5 = 18 \wedge 13 - 5 = 8),$$

$$10^2 - 3^2 = 13 \cdot 7 \Rightarrow (13 + 7 = 20 \wedge 13 - 7 = 6),$$

$$10^2 - 7^2 = 17 \cdot 3 \Rightarrow (17 + 3 = 20 \wedge 17 - 3 = 14),$$

$$11^2 - 6^2 = 17 \cdot 5 \Rightarrow (17 + 5 = 22 \wedge 17 - 5 = 12),$$

$$11^2 - 8^2 = 19 \cdot 3 \Rightarrow (19 + 3 = 22 \wedge 19 - 3 = 16),$$

$$12^2 - 5^2 = 17 \cdot 7 \Rightarrow (17 + 7 = 24 \wedge 17 - 7 = 10),$$

$$12^2 - 7^2 = 19 \cdot 5 \Rightarrow (19 + 5 = 24 \wedge 19 - 5 = 14),$$

$$13^2 - 6^2 = 19 \cdot 7 \Rightarrow (19 + 7 = 26 \wedge 19 - 7 = 12),$$

$$13^2 - 10^2 = 23 \cdot 3 \Rightarrow (23 + 3 = 26 \wedge 23 - 3 = 20),$$

$$14^2 - 3^2 = 17 \cdot 11 \Rightarrow (17 + 11 = 28 \wedge 17 - 11 = 6),$$

$$14^2 - 9^2 = 23 \cdot 5 \Rightarrow (23 + 5 = 28 \wedge 23 - 5 = 18),$$

$$15^2 - 2^2 = 17 \cdot 13 \Rightarrow (17 + 13 = 30 \wedge 17 - 13 = 4),$$

$$15^2 - 4^2 = 19 \cdot 11 \Rightarrow (19 + 11 = 30 \wedge 19 - 11 = 8),$$

$$15^2 - 8^2 = 23 \cdot 7 \Rightarrow (23 + 7 = 30 \wedge 23 - 7 = 16),$$

$$16^2 - 3^2 = 19 \cdot 13 \Rightarrow (19 + 13 = 32 \wedge 19 - 13 = 6),$$

$$17^2 - 6^2 = 23 \cdot 11 \Rightarrow (23 + 11 = 34 \wedge 23 - 11 = 12),$$

$$17^2 - 12^2 = 29 \cdot 5 \Rightarrow (29 + 5 = 34 \wedge 29 - 5 = 24),$$

$$17^2 - 14^2 = 31 \cdot 3 \Rightarrow (31 + 3 = 34 \wedge 31 - 3 = 28),$$

$$18^2 - 5^2 = 23 \cdot 13 \Rightarrow (23 + 13 = 36 \wedge 23 - 13 = 10),$$

$$18^2 - 11^2 = 29 \cdot 7 \Rightarrow (29 + 7 = 36 \wedge 29 - 7 = 22),$$

$$18^2 - 13^2 = 31 \cdot 5 \Rightarrow (31 + 5 = 36 \wedge 31 - 5 = 26),$$

$$19^2 - 12^2 = 31 \cdot 7 \Rightarrow (31 + 7 = 38 \wedge 31 - 7 = 24),$$

$$20^2 - 3^2 = 23 \cdot 17 \Rightarrow (23 + 17 = 40 \wedge 23 - 17 = 6),$$

$$20^2 - 17^2 = 37 \cdot 3 \Rightarrow (37 + 3 = 40 \wedge 37 - 3 = 34),$$

$$\dots .$$

This is the proof.

V. Supplement

Theorem 5. *For all* $n \in \{3,5,7,\dots\}$ *and for all* $z \in \{3,7,11,\dots\}$ *the equation* $z^n = u^2 + v^2$ *has no primitive solutions.*

Proof. Suppose that for some $n \in \{3,5,7,\dots\}$ and for some $z \in \{3,7,11,\dots\}$: $z^n = u^2 + v^2$. The numbers z, u and v are coprime and odd $u - v > 0$.

On the strength of the **Theorem 1** we get –

For some $n \in \{3,5,7,\dots\}$ and for some $z \in \{3,7,11,\dots\}$ and for some $d, k \in \{1,3,5,7,9,\dots\}$ and for some and for some $s, u, v \in \{1,2,3,\dots\}$ such that $u - v$ is odd and $k > 2s$:

$$\left[\left(\frac{z^n + d^2}{2d}\right)^2 = \left(\frac{2k+1+4s+1}{2d}\right)^2 = \left(\frac{k+2s+1}{d}\right)^2 \right.$$
$$= u^2 + \left(\frac{z^n - d^2}{2d}\right)^2 + v^2 \wedge \frac{z^n - d^2}{2d}$$
$$\left. = \frac{2k+1-4s-1}{2d} = \frac{k-2s}{d}\right] \in \mathbf{0},$$

inasmuch as

$$[4 \mid (k+2s+1)^2 \wedge 4 \nmid u^2 + (k-2s)^2 + v^2]. \;\square$$

Golden Nyambuya proved reputedly that – For all $n \in \{3,5,7,\dots\}$ the equation $z^n = u^2 + v^2$ has no primitive solutions in $\{1,2,3,\dots\}$ with $z \in \{3,5,7,\dots\} - \{3^2, 5^2, 7^2, \dots\}$. [7]

Corollary 1. For some $n \in \{3,5,7,\dots\}$ and for some $z \in \{5,9,13,\dots\}$ and for some relatively prime natural numbers u, v such that $u - v$ is positive and odd:

$$z^n = u^2 + v^2 \Longrightarrow (u^2 - v^2, 2uv, u^2 + v^2).$$

Example 1.

$$5^3 = 11^2 + 2^2 \Longrightarrow (11^2 + 2^2, 44, 11^2 + 2^2).$$

Example 2.

$$17^3 = 52^2 + 47^2 \Longrightarrow (52^2 - 47^2, 4888, 52^2 + 47^2).$$

Example 3.

$$29^3 = 145^2 + 58^2$$
$$\Longrightarrow (145^2 - 58^2, 16820, 145^2 + 58^2).$$

This is the corollary. This is the supplement.

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