A normal hyperbolic, global extension of the Kerr metric

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Abstract

A restriction of the Boyer-Lindquist model of the Kerr metric is considered that is globally hyperbolic on the space manifold $\mathbf{R}^3$ with the origin excluded, the quotient m/r being unrestricted. The model becomes in this process a generalization of Brillouin's model, describing the gravitational field of a rotating massive point particle.

1 Preliminary warnings

i) In this paper "Extension of a metric" has a more restricted meaning than that that is usual; ii) Besides its usual meaning the word "Singularity" means also the break of the hyperbolic type of the space-time metric. Signatures +2 and -2 are not accepted to co-exist in a space-time model.

2 Weyl-like description of stationnary space-time models

A Weyl-like description of a stationnary space-time model, [3]-[9], is a rewriting of a general metric:

$$ds^2 = g_{44}(x^k)dt^2 + 2g_{4i}(x^k)dtdx^i + g_{ij}(x^k)dx^idx^j, \quad i,j = 1,2,3 \tag{1}$$

as:

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$$ds^2 = -A^2(-dt + f_i dx^i)^2 + A^{-2} d\bar{s}^2, \tag{2}$$

with:

$$A^2 = -g_{44}, \quad f_i = A^{-2} g_{4i}, \quad \bar{g}_{ij} = A^2 g_{ij} + g_{i4} g_{j4} \tag{3}$$

being understood from the beginning that ds^2 is an acceptable local space-time model only on those domains of the variables x^i where A is real and $d\bar{s}^2$ is a positive definite proper Riemannian metric.

3 The Boyer-Lindquist coordinates

Using Boyer-Lindquist coordinates r, θ, ϕ, [2], named spherical polar coordinates, the coefficients of ds^2 are:

$$\begin{aligned}
g_{44} &= -1 + \frac{2mr}{r^2 + a^2 \cos^2\theta} && (4)\\
g_{34} &= \frac{2mra \sin^2\theta}{r^2 + a^2 \cos^2\theta} && (5)\\
g_{33} &= \left(r^2 + a^2 + \frac{2mra^2 \sin^2\theta}{r^2 + a^2 \sin^2\theta} \right) \sin^2\theta && (6)\\
g_{11} &= \frac{r^2 + a^2 \cos^2\theta}{a^2 - 2mr + r^2} && (7)\\
g_{22} &= r^2 + a^2 \cos^2\theta && (8)
\end{aligned}$$

or equivalently:

$$A^2 = 1 - \frac{2mr}{r^2 + a^2 \cos^2\theta}, \quad f_3 = A^{-2} g_{43} \tag{9}$$

and:

$$\begin{aligned}
\bar{g}_{11} &= 1 - \frac{a^2 \sin^2\theta}{a^2 - 2mr + r^2} && (10)\\
\bar{g}_{22} &= a^2 \cos^2\theta - 2mr + r^2 && (11)\\
\bar{g}_{33} &= (a^2 - 2mr + r^2) \sin^2\theta && (12)
\end{aligned}$$

The remaining components being zero. m is the mass parameter and a is the angular momentum parameter.

Assuming that $a = m = 0$ $d\bar{s}^2$ becomes the Euclidean metric:

$$d\tilde{s}^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{13}$$

There is no problem identifying θ and ϕ as polar angles. But this does not authorizes to identify r with the corresponding radial distance, because any radial coordinate transformation:

$$r \rightharpoonup \psi(r,\theta;a,m) \text{ such that } \psi(r,\theta;0,0) = r \tag{14}$$

would change the meaning of r. The next section illustrates this assertion.

4 A global extension of the Kerr metric

A^2 becomes zero in two circumstances that correspond to two singular points of the metric . Namely when the r coordinate is:

$$r^{\pm} = m \pm \mu \tag{15}$$

with:

$$\mu = \sqrt{m^2 - a^2\cos^2\theta} \tag{16}$$

Since $r^+ > r^-$ and both are positive this means that the Kerr metric is globally hyperbolic with positive values of r only in the interval $[r^+, \infty]$, r^+ being a function of θ and the two parameters m and a. In this domain only the Kerr metric is a legitimate space-time relativistic model.

Let us consider the coordinate transformation:

$$r \to r + r^+ = r + m + \mu \tag{17}$$

that it is a legitimate one only if $m \geq a$. The new values of the variables A, f_3, and $\bar{g}_{ij}$ are:

$$g_{44} = -1 + \frac{2m\sigma}{\sigma^2 + a^2\cos^2\theta} \tag{18}$$

$$g_{11} = \frac{\sigma^2 + a^2\cos^2\theta}{(a^2 - 2m\sigma + \sigma^2} \tag{19}$$

$$g_{22} = \frac{\sigma^2 + a^2\cos^2\theta)a^4\cos^2\theta\sin^2\theta}{\mu^2(a^2 - 2m\sigma + \sigma^2)} \tag{20}$$

$$g_{33} = \left(\sigma^2 + a^2 + \frac{2m\sigma a^2\sin^2\theta}{\sigma^2 + a^2\cos^2\theta}\right)\sin^2\theta \tag{21}$$

$$g_{34} = \frac{2m\sigma a\sin^2\theta}{\sigma^2 + a^2\cos^2(\theta)} \tag{22}$$

$$g_{12} = \frac{(\sigma^2 + a^2\cos^2\theta)a^2\cos\theta\sin\theta}{(a^2 - 2m\sigma + \sigma^2)\mu} \tag{23}$$

with the notation simplification:

$$\sigma \equiv r + m + \mu \tag{24}$$

Equivalently:

$$A^2 = -g_{44}, \quad f_3 = -\frac{2m(r+m+\mu)a\sin^2\theta}{2\mu r + r^2} \tag{25}$$

and:

$$\bar{g}_{11} = \frac{2\mu r + r^2}{a^2\sin^2\theta + 2r\mu + r^2} \tag{26}$$

$$\bar{g}_{22} = \frac{(a^2(m^2+r^2)\sin^2\theta + (m^2-a^2)r^2 + 2\mu^3 r)(2\mu r + r^2)}{\mu^2(2r\mu + r^2 + a^2\sin^2\theta)} \tag{27}$$

$$\bar{g}_{33} = (a^2\sin^2\theta + 2\mu r + r^2)\sin^2\theta \tag{28}$$

$$\bar{g}_{12} = \frac{a^2\cos\theta\sin\theta(2\mu r + r^2)}{(a^2\sin^2\theta + 2r\mu + r^2)\mu} \tag{29}$$

The two interesting limit cases of the preceding line-element are the Brillouin line-element [10]-[14] corresponding to $a = 0$ and that corresponding to $a = m$ that is the maximum value that a can have.

5 The Weyl space-like metric

The Weyl space-like metric $d\bar{s}^2$ with (26)-(29) can easily be written in diagonal form:

$$d\bar{s}^2 = \bar{g}_{11}\left(dr + \frac{\bar{g}_{12}}{\bar{g}_{11}}d\theta\right)^2 + \frac{1}{\bar{g}_{11}}(\bar{g}_{11}\bar{g}_{22} - \bar{g}_{12}^2)d\theta^2 + \bar{g}_{33}d\phi^2 \tag{30}$$

From inspection of (49) we see that $\bar{g}_{ii}$, $i = 1, 2, 3$, are definite positive and it is easy to prove that:

$$\bar{g}_{11}\bar{g}_{22} - \bar{g}_{12}^2 = \frac{r^2(2\mu + r)^2(a^2\sin^2\theta + r^2 + 2\mu r)}{a^4\sin^4\theta + 2a^2 r(r + 2\mu)\sin^2\theta + r^3(r + 4\mu)} \tag{31}$$

Therefore $d\bar{s}^2$ is a proper Riemannian metric.

It is the relationship between this metric and the Euclidean metric (13) that allows to calculate the optical length of an optic fiber whatever the circuit that is considered.

6 Polar plots

The first figure below is the polar-plot of the singular line $g_{44} = 0$, assuming that $a = m = 1$, when using the Boyer-Lindquist coordinates (red graph). Notice that θ being a polar angle its value is constrained to be in the closed interval $[0, \pi]$ and therefore only the upper part of the plot is relevant.

Using the global coordinates of this paper the graph is reduced to the center of the plot. This makes unambiguous the choice of r, frozens its interpretation as the distance from the singular source of the point of space being considered and selects the interval $]0, \infty[$ as the interval where $d\bar{s}^2$ is a positive definite metric.

7 Circular orbits

I consider in this section the geodesics whose space orbits are circles on the equator plane of symmetry $\theta = 0$ and $\Omega = d\phi/dt$ is constant. The relevant non zero, Christoffel symbols to take into account are:

$$\Gamma^3_{41} = -\frac{ma}{(r+m)^2(a^2+2mr+r^2)}, \quad \Gamma^3_{31} = -\frac{ma^2 - 4m^2r - 4mr^2 - r^3)}{(r+m)^2(a^2+2mr+r^2)} \tag{32}$$

The algebraic equation to be solved is:

$$2(a^2m - 8m^3 - 12m^2r - 6m^2r^2 - r^3)\Omega^2 + 2ma\Omega + m = 0 \tag{33}$$

and leads easily to the following angular velocity solutions:

$$\Omega^{\pm} = \frac{-ma \pm \sqrt{m(2m+r)^3}}{a^2m - 8m^3 - 12m^2r - 6mr^2 - r^3} \tag{34}$$

Consider the following quantity:

$$Ge(\Omega^{\pm}) = g_{44} + 2g_{34}\Omega^{\pm} + g_{33}(\Omega^{\pm})^2) \tag{35}$$

If Ge is negative the circular orbit is time-like. If it is zero it is light-like. Otherwise it is space-like.

The green graph of the second figure below is that of $G_e^+ = Ge^-$ corresponding to the Brillouin solution, $a = 0$, assuming that $m = 1$. In this case there is a single light-like geodesic satisfying the required conditions at $r = 1/2$. Below this value the corresponding geodesics are space-like, i.e. tachyon ones.

The graphs red and blue are the those of Ge^+ and Ge^- corresponding to the extreme Kerr model with $m = a = 1$. In this second case while for one direction of rotation all circular geodesic orbits are time-like. For the other rotation direction there are both time-like and space-like circular geodesic orbits separated by by a light one at $r \cong 1.292063116$.

The two main results of this paper are: i) The first postulate of General relativity requiring that a space-time model should be globally hyperbolic on $\mathbf{R}^3$ leaves out of physics such phantasies as "black or white holes"; ii) Surprisingly it appears that geodesic space-like trajectories appear quite naturally in Kerr model, and if it turns out that this is a general feature of many other solutions it might be a clue to why tachyons come out also naturally in some quantum gravity theories.

8 Rotation

Kerr's metric is a stationary exterior solution of Einstein's field equations such that de rotational of the vector with components $f_1 = f_2 = 0$ and f_3 as in (25) is not zero:

$$\Omega_{ij} \equiv \partial_i f_j - \partial_j f_i \neq 0 \tag{36}$$

and this means that there is an infinitesimal rotation at each event.

On the other hand it is possible to introduce a connected concept of rotation which is more specific and somewhat more global in the sense that it is more what we mean when saying that an extended body rotates with respect to an axis of symmetry. This new concept though is not about the rotation of the source itself, that it is a single point in this paper, but about the contribution of the rotation of the source on the exterior field. In this sense rotation is a concept to be defined.

Let us consider the Kerr-Brillouin's line-element defined by the potentials (18-(22):

$$dg^2 = g_{44}dt^2 + g_{33}d\varphi^2 + g_{34}d\varphi dt + g_{11}dr^2 + g_{22}d\theta^2 \tag{37}$$

and let me introduce, to be used for comparison, the line element dl^2 obtained from the preceding one by two substitutions:

$$a \to 0, \quad d\varphi \to d\varphi + \omega dt \tag{38}$$

What we get is the Schwarzshild-Brillouin's line-element forced to rotate with constant angular rotation ω in the most elementary meaning.

$$dl^2 = l_{44}dt^2 + l_{33}d\varphi^2 + l_{34}d\varphi dt + l_{11}dr^2 + l_{22}d\theta^2 \tag{39}$$

where in particular:

$$l_{44} = \frac{(8m^3 + 12m^2r + 6mr^2 + r^3)\sin^2\theta\omega^2}{r + 2m} - \frac{r}{r + 2m} \tag{40}$$

and:

$$l_{34} = \omega(4m^2 + 4mr + r^2)\sin^2\theta \tag{41}$$

Let us view the space exterior to the point-source as the collection of all parallel circles that are at a distance r and a co-latitude θ, each with its line element derived from (37) with the contractions $dr = 0$, $d\theta = 0$. Let us also view he same collection of parallel circles with the line-element derived from (39) and consider now the mapping that to each circle with parameters r, θ and line-element (37) corresponds the same circle with line-element (39). Requiring the equation:

$$\frac{g_{34}}{g_{44}} = \frac{l_{34}}{l_{44}} \tag{42}$$

i.e.:

$$-\frac{2m(r + m + \mu)a\sin^2\theta}{2\mu r + r^2} = \frac{r + 2m)^3\sin^2\theta\omega}{\omega^2\sin^2\theta(8m^3 + 12m^2r + 6mr^2 + r^3) - r} \tag{43}$$

that is invariant under time-transformations $t \to t + \psi(r, t)$, introduces a relationship between the parameters a and ω that gives an interpretation of a as the generator of a rotation $\omega(r, \theta)$ depending on the parallel circle considered. This comes as follows:

solving the second order algebraic equation for ω we get:

$$\omega^{\pm} = \frac{1}{2}\frac{p \pm \sqrt{p^2 + 4\sin^2\theta f^2 qr}}{\sin^2\theta q} \tag{44}$$

where:

$$p = \sin^2\theta(4m^2 + 4mr + r^2)(r + 2 * m) \tag{45}$$

$$q = 8m^3 + 12m^2r + 6mr^2 + r^3 \tag{46}$$

$$f = \frac{2\sin^2\theta am(r + m + \mu)}{-2\mu r - r^2} \tag{47}$$

The behaviours for $r \to \infty$ are:

$$\omega^{+} = -\frac{1}{2}\frac{r}{am\sin^2\theta} + O(1) \tag{48}$$

and:

$$\omega^{-} = \frac{2am(-r+\mu+5m)}{r^4} \tag{49}$$

and therefore only ω^{-} is acceptable to establish a correspondence between ω and a.

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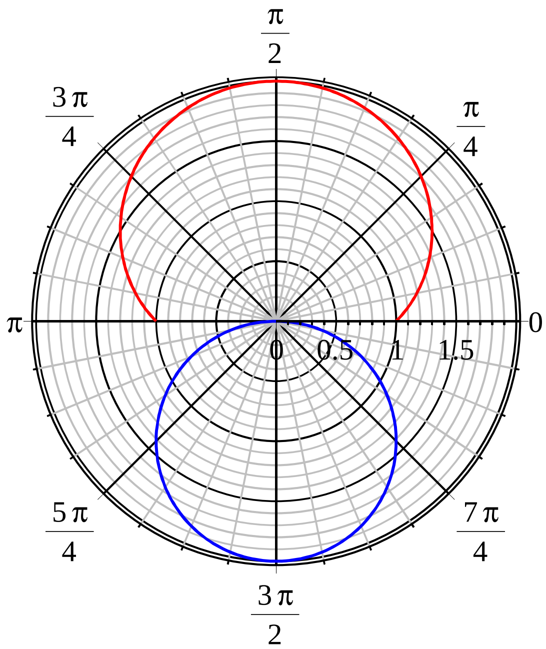

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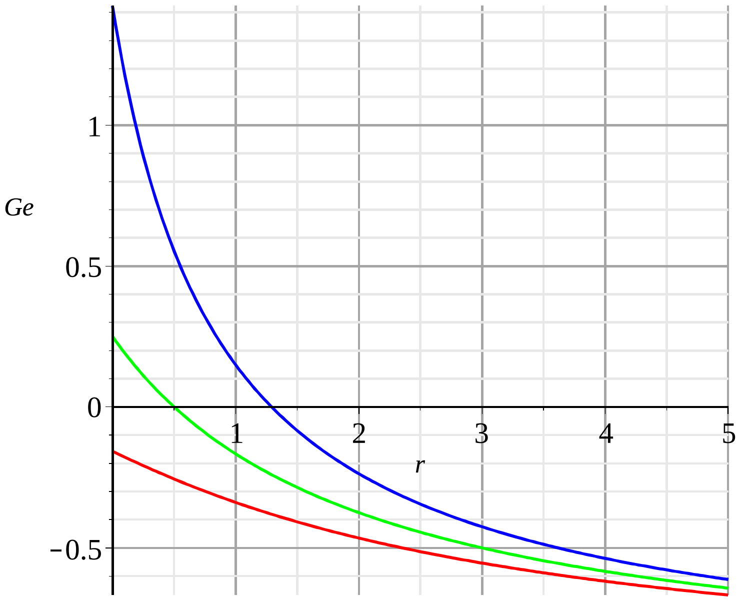

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