A normal hyperbolic, global extension of the Kerr metric

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Abstract

A restriction of the Boyer-Linquist model of the Kerr metric is considered that is globally hyperbolic on the space manifold $\mathbf{R}^3$ with the origin excluded, the quotient m/r being unrestricted. The model becomes in this process a generalization of Brillouin's model, describing the gravitational field of a rotating massive point particle.

1 Preliminary warnings

i) In this paper "Extension of a metric" has a more restricted meaning than that that is usual; ii) Besides its usual meaning the word "Singularity" means also the break of the hyperbolic type of the space-time metric. Signatures +2 and -2 are not accepted to co-exist in a space-time model.

2 Weyl-like description of stationnary space-time models

A Weyl-like description of a stationnary space-time model, [3]-[9], is a rewriting of a general metric:

$$ds^2 = g_{44}(x^k)dt^2 + 2g_{4i}(x^k)dtdx^i + g_{ij}(x^k)dx^idx^j, \quad i,j = 1,2,3 \qquad (1)$$

as:

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$$ds^2 = -A^2(-dt + f_i dx^i)^2 + A^{-2} d\bar{s}^2, \tag{2}$$

with:

$$A^2 = -g_{44}, \quad f_i = A^{-2} g_{4i}, \quad \bar{g}_{ij} = A^{-2} g_{ij} + g_{i4} g_{j4} \tag{3}$$

being understood from the beginning that ds^2 is an acceptable local space-time model only on those domains of the variables x^i where A is real and $d\bar{s}^2$ is a positive definite proper Riemannian metric.

3 The Boyer-Linquist coordinates

Using Boyer-Linquist coordinates r, θ, ϕ, [2], named spherical polar coordinates, the coefficients of ds^2 are:

$$\begin{aligned}
g_{44} &= -1 + \frac{2mr}{r^2 + a^2 \cos^2\theta} && (4)\\
g_{34} &= \frac{2mra \sin^2\theta}{r^2 + a^2 \cos^2\theta} && (5)\\
g_{33} &= \left(r^2 + a^2 + \frac{2mra^2 \sin^2\theta}{r^2 + a^2 \sin^2\theta}\right) \sin^2\theta && (6)\\
g_{11} &= \frac{r^2 + a^2 \cos^2\theta}{a^2 - 2mr + r^2} && (7)\\
g_{22} &= r^2 + a^2 \cos^2\theta && (8)
\end{aligned}$$

or equivalently:

$$A^2 = 1 - \frac{2mr}{r^2 + a^2 \cos^2\theta}, \quad f_3 = A^{-2} g_{43} \tag{9}$$

and:

$$\begin{aligned}
\bar{g}_{11} &= 1 - \frac{a^2 \sin^2\theta}{a^2 - 2mr + r^2} && (10)\\
\bar{g}_{22} &= a^2 \cos^2\theta - 2mr + r^2 && (11)\\
\bar{g}_{33} &= (a^2 - 2mr + r^2) \sin^2\theta && (12)\\
\bar{g}_{12} &= \frac{a^2 \cos\theta \sin\theta (2\mu r + r^2)}{(a^2 \sin^2\theta + 2r\mu + r^2)\mu} && (13)
\end{aligned}$$

where m is the mass parameter, a with $a^2 \leq m^2$, is the angular momentum parameter, and:

$$\mu = \sqrt{m^2 - a^2\cos^2\theta} \tag{14}$$

Notice that while there is no problem considering the limit $m \to 0$ of ds^2, this is not the case with $d\bar{s}^2$.

Assuming that $a = m = 0$ $d\bar{s}^2$ becomes the Euclidean metric:

$$d\tilde{s}^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{15}$$

there is no problem identifying θ and ϕ as polar angles. But this does not authorizes to identify r with the corresponding radial distance, because any radial coordinate transformation:

$$r \rightharpoonup \psi(r, \theta; a, m) \text{ such that } \psi(r, \theta; 0, 0) = r \tag{16}$$

would change the meaning of r. The next section illustrates this assertion.

4 A global extension of the Kerr metric

A^2 becomes zero in two circumstances that correspond to two singular points of the metric . Namely when the r coordinate is:

$$r^{\pm} = m \pm \mu \tag{17}$$

Since $r^+ > r^-$ and both are positive this means that the Kerr metric is globally hyperbolic with positive values of r only in the interval $[r^+, \infty]$, r^+ being a function of θ and the two parameters m and a. In this domain only the Kerr metric is a legitimate space-time relativistic model.

Let us consider the coordinate transformation:

$$r- > r + r^+ = r + m + \sqrt{m^2 - a^2 cos^2\theta} \tag{18}$$

the new values of the variables A, f_3, and $\bar{g}_{ij}$ are:

$$g_{44} = -1 + \frac{2m\sigma}{\sigma^2 + a^2\cos^2\theta} \tag{19}$$

$$g_{11} = \frac{\sigma^2 + a^2\cos^2\theta}{(a^2 - 2m\sigma + \sigma^2} \tag{20}$$

$$g_{22} = \frac{\sigma^2 + a^2\cos^2\theta)a^4\cos^2\theta\sin^2\theta}{\mu^2(a^2 - 2m\sigma + \sigma^2)} \tag{21}$$

$$g_{33} = \left(\sigma^2 + a^2 + \frac{2m\sigma a^2\sin^2\theta}{\sigma^2 + a^2\cos^2\theta}\right)\sin^2\theta \tag{22}$$

$$
\begin{aligned}
g_{34} &= \frac{2m\sigma a\sin^2\theta}{\sigma^2+a^2\cos^2(\theta)} && (23)\\
g_{12} &= \frac{(\sigma^2+a^2\cos^2\theta)a^2\cos\theta\sin\theta}{(a^2-2m\sigma+\sigma^2)\mu} && (24)
\end{aligned}
$$

with the the notation simplification:

$$
\sigma \equiv r+m+\mu \tag{25}
$$

Equivalently:

$$
A^2=-g_{44}, \quad f_3=-\frac{2m(r+m+\mu)a\sin^2\theta}{2\mu r+r^2} \tag{26}
$$

and:

$$
\begin{aligned}
\bar{g}_{11} &= \frac{2\mu r+r^2}{a^2\sin^2\theta+2r\mu+r^2} && (27)\\
\bar{g}_{22} &= \frac{(a^2(m^2+r^2)\sin^2\theta+(m^2-a^2)r^2+2\mu^3 r)(2\mu r+r^2)}{\mu^2(2r\mu+r^2+a^2\sin^2\theta)} && (28)\\
\bar{g}_{33} &= (a^2\sin^2\theta+2\mu r+r^2)\sin^2\theta && (29)\\
\bar{g}_{12} &= \frac{a^2\cos\theta\sin\theta(2\mu r+r^2)}{(a^2\sin^2\theta+2r\mu+r^2)\mu} && (30)
\end{aligned}
$$

The two interesting limit cases of the preceding line-element are the Brillouin line-element [10]-[14] corresponding to $a=0$ and that corresponding to $a=m$ that is the maximum value that a can have.

5 The Weyl space-like metric

The Weyl space-like metric $d\bar{s}^2$ with (36)-(36) can easily be written in diagonal form:

$$
d\bar{s}^2=\bar{g}_{11}\left(dr+\frac{\bar{g}_{12}}{\bar{g}_{11}}d\theta\right)^2+\frac{1}{\bar{g}_{11}}(\bar{g}_{11}\bar{g}_{22}-\bar{g}_{12}^2)d\theta^2+\bar{g}_{33}d\phi^2 \tag{31}
$$

From inspection of (36) we see that $\bar{g}_{ii}$, $i=1,2,3$, are definite positive and it is easy to prove that:

$$
\bar{g}_{11}\bar{g}_{22}-\bar{g}_{12}^2=\frac{r^2(2\mu+r)^2(a^2\sin^2\theta+r^2+2\mu r)}{a^4\sin^4\theta+2a^2r(r+2\mu)\sin^2\theta+r^3(r+4\mu)} \tag{32}
$$

Therefore $d\bar{s}^2$ is a proper Riemannian metric.

It is the relationship between this metric and the Euclidean metric (15) that allows to calculate the optical length of an optic fiber whatever the circuit that is considered.

6 Polar plots

The first figure below is the polar-plot of the singular line $g_{44} = 0$, assuming that $a = m = 1$, when using the Boyer-Linquist coordinates (red graph). Notice that θ being a polar angle its value is constrained to be in the closed interval $[0, \pi]$ and therefore only the upper part of the plot is relevant.

Using the global coordinates of this paper the graph is reduced to the center of the plot. This makes unambiguous the choice of r, frozens its interpretation as the distance from the singular source of the point of space being considered and selects the interval $]0, \infty[$ as the interval where $d\bar{s}^2$ is a positive definite metric.

7 Circular orbits

I consider in this section the geodesics whose space orbits are circles on the equator plane of symmetry $\theta = 0$ and $\Omega = d\phi/dt$ is constant. The relevant non zero, Christoffel symbols to take into account are:

$$\Gamma^3_{41} = -\frac{ma}{(r+m)^2(a^2+2mr+r^2)}, \quad \Gamma^3_{31} = -\frac{ma^2 - 4m^2r - 4mr^2 - r^3)}{(r+m)^2(a^2+2mr+r^2)} \tag{33}$$

The algebraic equation to be solved is:

$$2(a^2m - 8m^3 - 12m^2r - 6m^2r^2 - r^3)\Omega^2 + 2ma\Omega + m = 0 \tag{34}$$

and leads easily to the following angular velocity solutions:

$$\Omega^{\pm} = \frac{-ma \pm \sqrt{m(2m+r)^3}}{a^2m - 8m^3 - 12m^2r - 6mr^2 - r^3} \tag{35}$$

Consider the following quantity:

$$Ge(\Omega^{\pm}) = g_{44} + 2g_{34}\Omega^{\pm} + g_{33}(\Omega^{\pm})^2) \tag{36}$$

If Ge is negative the circular orbit is time-like. If it is zero it is light-like. Otherwise it is space-like.

The green graph of the second figure below is that of $G_e^+ = Ge^-$ corresponding to the Brillouin solution, $a = 0$, assuming that $m = 1$. In this case there is a single light-like geodesic satisfying the required conditions at $r = 1/2$. Below this value the corresponding geodesics are space-like, i.e. tachyon ones.

The graphs red and blue are the those of Ge^+ and Ge^- corresponding to the extreme Kerr model with $m = a = 1$. In this second case while for one direction of rotation all circular geodesic orbits are time-like. For the other rotation direction there are both time-like and space-like circular geodesic orbits separated by by a light one at $r \cong 1.292063116$.

The two main results of this paper are: i) The first postulate of General relativity requiring that a space-time model should be globally hyperbolic on $\mathbf{R}^3$ leaves out of physics such phantasies as "black or white holes"; ii) Surprisingly it appears that geodesic space-like trajectories appear quite naturally in Kerr model, and if it turns out that this is a general feature of many other solutions it might be a clue to why tachyons come out also naturally in some quantum gravity theories.

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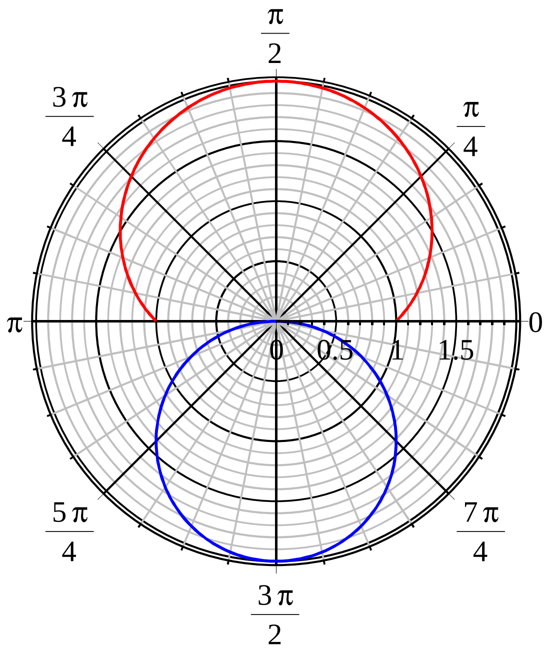
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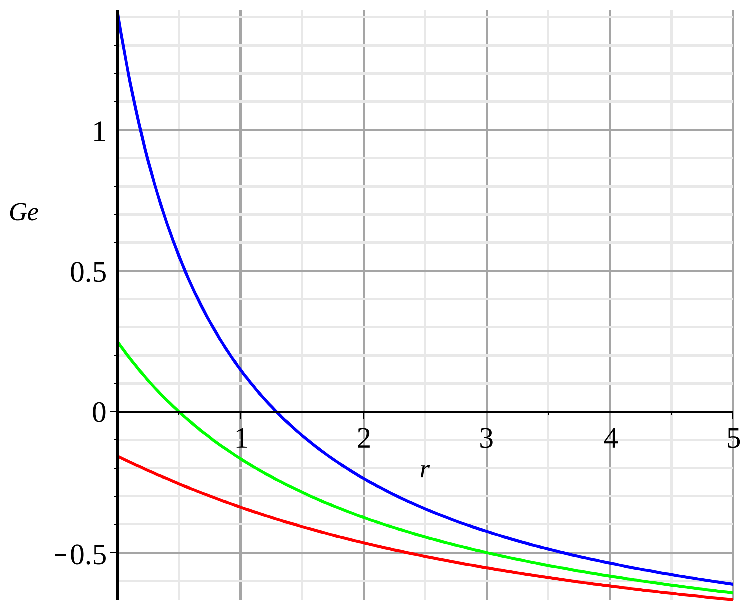
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