Cantor's Diagonal Argument Reexamined

Richard L. Hudson 11-28-2017

abstract

This analysis shows Cantor's diagonal argument cannot form a new sequence that is not a member of a complete list.

1. the argument

Assume a complete list L of random infinite sequences. Each sequence s is a unique infinite pattern of symbols from the set {0, 1}. A sample of a random list begins as:

100101...
010011...
110110...
100000...
000111...
111001...

A sequence p is formed from the diagonal elements (underlined) by applying the rule, if 0 then 1 else 0, to each position from left to right. The diagonal d=110011... is transformed via the substitution rule to the horizontal p = 001100...

1.1 Cantor's conclusion

Since p differs from each s in the sample by construction, it will differ from all s in the list L, therefore a new sequence p will be formed not in the list L. The set of integers N is not sufficient to count the list L. [1]

2. binary tree

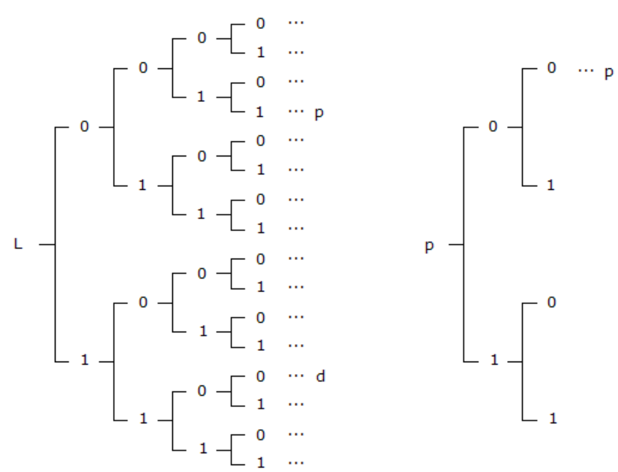

fig.1 fig.2

The binary tree (fig.1) shows the beginning of all possible sequences, each corresponding to a unique path from left to right. All s must begin with 0 or 1, thus all would be contained in the tree if extended without limit. The tree therefore is a representation of L as defined in par.1. Sequence p and its complement d are included in L as noted. Fig.2 is an enlarged view to extend p while avoiding crowding of the detail. The tree is symmetrical relative to a horizontal line through L. If the tree is rotated 180° on the line, the symbols 0 and 1 are interchanged showing the pair of sequences d and p are complementary and mirror images.
If all symbols 0 and 1 are interchanged in the list L, equivalent to a 180° rotation, the sample list is:

011010...
101100...
001001...
011111...
111000...
000110...

A sequence p is formed from the diagonal elements (underlined) as they appear, producing p = 001100..., the same as done in par.1. The transformation rule used in par.1 is therefore redundant. Additionally, a diagonal sequence has the same k positions and is formed from the same set of symbols {0, 1}, as those in the sample list, independently of direction.

3. finite sequences

Here is a program to form finite sequences, with only the essential operations.

```
1. i=0, k=position count     ;set parameters
2. i+1                       ;increment counter
3. rand(x)                   ;random selection of {0, 1}
4. print symbol
5.if i=k, do 7
6. do 2
7. stop
```

A random sequence of length 5, with $c=2^k=2^5=32$ combinations is 10110.

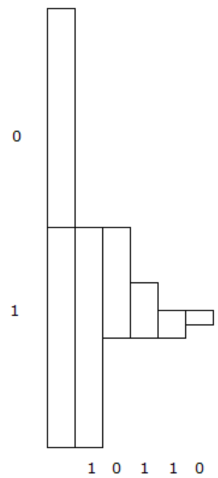

fig.3

In fig.3, the columns represent a subset of L, with the upper half L0 and the lower half L1.
With each program cycle, the selection eliminates half of the current subset of L. Since each s has to differ in only one position, comparing to all s in the list is also redundant.

4. infinite sequences

Here is a program to form infinite sequences, with only the essential operations.

```
1. i=0, k=position count      ;set parameters
2. i+1                        ;increment counter
3. rand(x)                    ;random selection of {0, 1}
4. print symbol
6. do 2
```

Redundant steps eliminated. Since there is no position k for comparison, step 5 will always be false, and step 7 will never occur.

A random partial sequence of length 5, with c indeterminate is 10110.

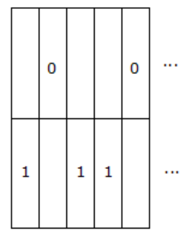

fig.4

As shown in fig.4, with each program cycle, the selection set of sequences remains constant equal to L, never ending in a single sequence. L contains itself in an infinite loop. This is equivalent to the construction in par.1 and 2.

5. uniqueness

For k=3, there are 8 combinations,

000	100
001	101
010	110
011	111

If an element is selected at random, 101, then compared to all members of the set, it will differ from all members except one, itself.
If an element is selected at random, 101, and removed, then compared to all remaining members of the set, it will differ from all those members.

conclusion

1. All sequences must begin with 0 or 1, therefore all sequences are an element of the tree.
2. Cantor is forming an existing random sequence in the tree. (par.2,4)
3. Cantor's conclusion, "it will differ from all s in the list L" (par.1.1), is false. In the only meaningful comparison, that of a set of 2^k members, p being a member and differing from all s is a contradiction. (par.5)
4. Processes require time. If the infinite sequence is always incomplete (par. 4), there are no s to list, and no diagonal to manipulate. The list L is empty.
5. That a list can be both complete and endless is a contradiction.
6. If the sequences could be produced, s_1, s_2, s_3, s_4,..., a corresponding integer could be assigned from N, with the assurance that N is inexhaustible.

references

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1. Cantor's Diagonal Argument, Wikipedia, Mar 2015