A simple proof for an extension of the Brouwer theorem.

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Abstract

A new proof for the Brouwer fix point theorem is given.

1 Proof

Given a manifold $\mathcal{M}$, its topological structure is fully encoded into the differentiable structure. Given that the differentiable structure is encoded in the bundle of vectorfields and the topological information in every vectorfield is given by its zero points and the number of positive, null, respectively negative number of eigenvalues of the Hessian (the symmetrization of the matrix of first partial derivatives of the vectorfield) at those points (because the latter is the only geometric invariant for any Riemannian connection) the Euler number (which is the only topological invariant of the whole manifold as such) must be written in the form

$$\chi(\mathcal{M}) = \sum_{i=0}^{n} (-1)^i N_{V,i}$$

where n is the dimension of the manifold and $N_{V,i}$ is the number of critical points with i positive eigenvalues in the Hessian. This formula would only work when the Hessian determines a level surface of codimension i which implies that the number of null critical eigenvalues need to be zero and that the vectorfield needs to be locally integrable. Those level surfaces are associated to dislocations responsible for the $n-i$'th betti number. This *has* to be given that the Betti numbers are additive under the connected sum of two manifolds and therefore $N_{V,i}$ is the only quantity available. Given that integrable vectorfields can be replaced by their opposite, Betti duality $b_{n-i} = b_i$ is an obvious fact. The factors $(-1)^i$ can be seen as part of the definition of the Euler number. Such formula would only make sense for vectorfields with a finite number of isolated zeroes, a subclass which is topologically distinguished indeed. This class is not characterized by the nonvanishing character of the determinant of the (symmetric part of the) Hessian; that one regards a smaller class which we need to consider here. From the topological point of view, not all vectorfields are equal and there is no further covariant characterization (which is independent of a geometry) of subbundles other than these regarding the number

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of zeroes and the aforementioned spectral characteristics of the Hessian. The entire vector bundle cannot serve by itself given that it is infinite dimensional whereas the topology is only finitely characterized and any vectorfield with a continuum number of zeroes would be useless too.

Weaker classes, such as the one with an isolated number of zeroes, do not satisfy this formula. For example,computation for a suitable vectorfield on the two dimensional disk and the three dimensional ball for an infinitesimal rotation gives

$$1 = 0$$

twice. Regarding nonintegrable vectorfields whose Hessian at the critical points is nonsingular, there is no objection as the level surfaces are still canonically defined; this provides for a slight extension of the usual result. This result can also be proved from the latter by means of a deformation argument. Consider now a diffeomorphism $\Phi : \mathcal{M} \to \mathcal{M}$, where $\mathcal{M}$ is a cobordism in the sense that it has nontrivial functions in that class, connected to the identity such that the "Hessian"

$$\frac{\partial \Phi^j}{\partial x^k}$$

is nonsingular. Then, regarding the associated vectorfield V,

$$\frac{\partial V^j}{\partial x^k}$$

is symmetric which means it is *globally* (due to the cobordism condition) associated to a function ψ. Hence, V has fixed points in case the Euler number is nonzero and therefore Φ has fixpoints. In particular, any n dimensional ball has Euler number one and therefore any "integrable" diffeomorphism connected to the identity has a fixpoint.