Energy momentum tensor and conservation of energy for two free charged spheres

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Abstract

We consider two charged spheres of the same charge freely moving along a fixed line. We use the energy momentum tensor to construct a moving surface through which there is no energy flow and show conservation of energy does not hold.

1 Energy flow surface

$\mathcal{S}_{AB}$ will be the following system. Let A and B, when at rest and infinitely far apart, be spheres with uniform charge and mass density. Let A have charge $Q > 0$, mass M, radius R and B have charge Q, mass m, and radius R. Position A and B so that their centres are always on the x_1 axis. Let A begin at negative x_1 infinity with velocity $u > 0$ and B begin at positive x_1 infinity with velocity $-u$. Let A and B move freely, that is, under the sole influence of the electromagnetic field of A and B.

Let $T^{\mu\nu}(t, \mathbf{x})$ be the energy momentum tensor for $\mathcal{S}_{AB}$. Define a vector field

$$\mathbf{v}(t, \mathbf{x}) = \left(\frac{T^{01}(t, \mathbf{x})}{T^{00}(t, \mathbf{x})}, \frac{T^{02}(t, \mathbf{x})}{T^{00}(t, \mathbf{x})}, \frac{T^{03}(t, \mathbf{x})}{T^{00}(t, \mathbf{x})}\right) \tag{1}$$

We can use the local existence and uniqueness theorem of ordinary differential equations[1] to define a unique vector valued function $\mathbf{x}_{x_2,x_3}(t) \in \mathbb{R}^3$ by

$$\dot{\mathbf{x}}_{x_2,x_3}(t) = \mathbf{v}\Big(\mathbf{x}_{x_2,x_3}(t), t\Big) \qquad \lim_{t\to-\infty} \mathbf{x}_{x_2,x_3}(t) = (0, x_2, x_3) \tag{2}$$

and for each t define a surface S_t of $\mathbb{R}^3$ by

$$S_t = \{\mathbf{x}_{x_2,x_3}(t) : x_2 \in \mathbb{R}, x_3 \in \mathbb{R}\} \tag{3}$$

We have S_t approaches the x_1 plane as $t \to -\infty$. The surface S_t divides $\mathbb{R}^3$ into two sets having S_t as intersection. One set is to the left of S_t and the other to the right. At time t let V_t be the set of points to the right of S_t.

2 New system of charges and conservation of energy

For system $\mathcal{S}_{AB}$ we have by conservation of energy

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{01}}{\partial x_1} + \frac{\partial T^{02}}{\partial x_2} + \frac{\partial T^{03}}{\partial x_3} = 0 \tag{4}$$

Now consider the following new system $\mathcal{S}_C$ of charges. For $\mathcal{S}_C$ let the charge density at $(t, \mathbf{x})$ be $\rho(t, \mathbf{x}) = T^{00}(t, \mathbf{x})$ and let the charges at point $(t, \mathbf{x})$ move with velocity $\mathbf{v}(t, \mathbf{x})$. The current density is then $\mathbf{J}(t, \mathbf{x}) = \rho(t, \mathbf{x})\mathbf{v}(t, \mathbf{x})$. We would have by (1) and (4) for $\mathcal{S}_C$ that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{5}$$

hence for $\mathcal{S}_C$ charge is conserved. A point $\mathbf{x}_{x_2,x_3}(t)$ of the surface S_t moves with velocity $\mathbf{v}(t, \mathbf{x}_{x_2,x_3}(t))$ and for $\mathcal{S}_C$ charges at $\mathbf{x}_{x_2,x_3}(t)$ move with velocity $\mathbf{v}(t, \mathbf{x}_{x_2,x_3}(t))$. Consequently no charge passes through the surface S_t. We can conclude from this and conservation of charge for $\mathcal{S}_C$ that the total charge to the right of S_t remains constant in time. That is the integral $\int_{V_t} \rho(t, \mathbf{x}) dx_1 dx_2 dx_3$ remains constant in time. Since $T^{00}(t, \mathbf{x}) = \rho(t, \mathbf{x})$ we have

$$\int_{V_t} T^{00}(t, \mathbf{x}) dx_1 dx_2 dx_3 = \int_{V_t} \rho(t, \mathbf{x}) dx_1 dx_2 dx_3 \tag{6}$$

Consequently for $\mathcal{S}_{AB}$ the amount of energy to the right of S_t remains constant in time.

3 Contradiction

Let $\mathcal{S}_B$ be the system consisting of just B moving with constant velocity u and no A. Define $E(Q, m, R, u)$ to be the total energy of $\mathcal{S}_B$.

For what follows the system will be $\mathcal{S}_{AB}$. Now S_t approaches the x_1 plane as $t \to -\infty$ and the distances between A, B, and S_t becomes infinity large as $t \to -\infty$. Consequently the amount of energy to the right of S_t approaches $E(Q, m, R, u)$ as $t \to -\infty$.

If $M = m$ then by symmetry at any time S_t will be the x_1 plane. If also u is not too large repulsion between A and B will cause the velocities of A and B to change direction after some time and the distances between A, B, and S_t to become infinitely large as $t \to \infty$. Consequently if M is approximately equal to m and u not too large then the velocities of A and B will change direction after some time and the distances between A, B, and S_t become infinitely large as $t \to \infty$. After the velocity of B changes direction B will then go to positive x_1 infinity. Let $w > 0$ be the velocity B has at infinity. Since the distances between A, B, and S_t become infinitely large as $t \to \infty$ the amount of energy to the right of S_t approaches $E(Q, m, R, w)$ as $t \to \infty$. We showed in the previous section the amount of energy to the right of S_t remains constant in time. Consequently the amount of energy to the right of S_t as $t \to -\infty$ is the same as the amount of energy to the right of S_t as $t \to \infty$ hence

$$E(Q, m, R, u) = E(Q, m, R, w) \tag{7}$$

Solving the equations of motion we find if $M > m$ then $u < w$. This contradicts (7) since

$$E(Q, m, R, u) < E(Q, m, R, w) \tag{8}$$

when $u < w$ hence conservation of energy does not hold.

References

[1] L. Loomis and S. Sternberg, *Advanced Calculus*, (Addison-Wesley, Reading, MA, 1968)