A three dimensions Navier-Stokes equations solution proposition

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Abstract

Here is proposed a solving Navier-Stokes equations three dimensions fluid model, described with cylindrical coordinates, composed of null radial and vertical velocities, and of a cross-radial velocity. Equations conditions verifications and calculus description leading to the expression of pressure are given. A description of the problem can be found here.

Proposition

We define the cross-radial velocity:

$$u_\theta = u_\theta^o \sqrt{\frac{r}{r_i}}\; e^{-\dfrac{r^2}{r_a^2} - \dfrac{z^2}{z_a^2}} \quad \text{where} \quad \begin{cases} u_\theta^o = u_\theta^o(x) = u_\theta(r,\theta,z,0) \text{ is the cross-radial component of the initial velocity } u^o(x), \\ \text{which is a divergence-free vector field } (\nabla \cdot u^o(x) = 0) \text{ satisfying the condition * below} \\ r_i, r_a^o, z_a^o \text{ are non null length constants} \\ r_a = \sqrt{\frac{\tau}{t+\tau_1} - \frac{\tau}{\tau_1} + r_a^{o2}},\; r_a^o = r_a(0) \\ z_a = \sqrt{\frac{\tau}{t+\tau_1} - \frac{\tau}{\tau_1} + z_a^{o2}},\; z_a^o = z_a(0) \\ t \text{ represents the time } (t \geq 0) \\ \tau \text{ and } \tau_1 \text{ are non null time constants} \\ \text{r is the radial coordinate} \\ \text{z is the vertical coordinate} \end{cases}$$

* $|\partial_x^\alpha u^o(x)| \leq C_{\alpha K}(1+|x|)^{-K}$ on $\mathbb{R}^3$ (x representing the position), for any $\alpha \in \mathbb{N}^3$ and $K > 0$, $C_{\alpha K}$ being a constant depending on α and K.

This fluid moves in three dimensions because the cross-radial velocity changes according to the radial distance r and the height z. So, this velocity evolves in these two dimensions, and by definition in the azimuth theta one. The movement so happens in the three dimensions.

We verify now the conditions of incompressibility (1) and bounded energy (2):

(1) Incompressibility is verified if div $u = 0$, where $u = (u_r, u_\theta, u_z)$, which gives:

div $u = 0 \Leftrightarrow \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \Leftrightarrow \frac{\partial u_\theta}{\partial \theta} = 0$, because radial and vertical velocities are null.

$u^o(x)$ being divergence-free, we have: $\frac{1}{r}\frac{\partial r u_r^o(x)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta^o(x)}{\partial \theta} + \frac{\partial u_z^o(x)}{\partial z} = 0$, implying $\frac{\partial u_\theta^o(x)}{\partial \theta} = 0$.

Radial and vertical velocities, $\frac{\partial u_\theta^o}{\partial \theta}$, being null, implying $\frac{\partial u_\theta}{\partial \theta} = \frac{\partial u_\theta^o}{\partial \theta}\sqrt{\frac{r}{r_i}e^{-\frac{r^2}{r_a^2} - \frac{z^2}{z_a^2}}}$ being also null, incompressibility is verified.

(2) The energy of the fluid remains finite if $\int_{\mathbb{R}^3} |u|^2\, r\, dr\, d\theta\, dz$ is upper-bounded.

$$\int_{\mathbb{R}^3} |u|^2\, r\, dr\, d\theta\, dz = \int_{\mathbb{R}^3} u_\theta^2\, r\, dr\, d\theta\, dz = \int_{\mathbb{R}^3} {u_\theta^o}^2 \frac{r^2}{r_i} e^{-\frac{2r^2}{r_a^2}}\, e^{-\frac{2z^2}{z_a^2}}\, dr d\theta dz$$

We know that : * $|\partial_x^\alpha u_\theta^o| \leq C_{\alpha K}(1+|x|)^{-K}$, for any $\alpha \in \mathbb{N}^3$ and $K > 0$.

So, for $\alpha = (0,0,0)$, $|u_\theta^o| \leq C_{0K}(1+|x|)^{-K}$, so ${u_\theta^o}^2 \leq {C_{0K}}^2(1+|x|)^{-2K} = {C_{0K}}^2(1+\sqrt{r^2+z^2})^{-2K} \leq {C_{0K}}^2$

So, we have:

$\int_{\mathbb{R}^3} u_\theta^2\, r\, dr\, d\theta\, dz \leq \int_{\mathbb{R}^3} {C_{0K}}^2 \frac{r^2}{r_i} e^{-\frac{2r^2}{r_a^2}}\, e^{-\frac{2z^2}{z_a^2}}\, dr d\theta dz$, which is a converging integral,

consequently upper-bounded. Bounded energy condition is also verified.

We now verify the existence of the pressure present in the corresponding Navier-Stokes momentum equation, in cylindrical coordinates, the force taken identically zero, following the Clay Math Institute problem definition:

$$r : \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} +$$

$$\nu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]$$

$$\theta : \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} +$$

$$\nu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \phi} - \frac{u_\theta}{r^2}\right]$$

$$z : \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = -\frac{\partial p}{\partial z}$$

$$+\nu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right]$$

Given the precedent informations, we can simplify these equations:

$$\frac{\partial p}{\partial r} = \frac{u_\theta^2}{r}$$

$$\frac{\partial p}{\partial \theta} = -r\frac{\partial u_\theta}{\partial t} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2}\right]$$

$$\frac{\partial p}{\partial z} = 0$$

Which gives, integrating $\frac{\partial p}{\partial r}$: $\quad P(r,\theta,z,t) = \sqrt{\frac{\pi}{8}}\, u_\theta^{o^2}\, \frac{r_a}{r_i}\, e^{-\frac{2z^2}{z_a^2}}\, \mathrm{erf}(\sqrt{2}\frac{r}{r_a}) \;+\; a(\theta,z)$

(erf designating the error function, $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$)

$$\frac{\partial p}{\partial \theta} = \frac{\partial a(\theta,z)}{\partial \theta} = -r\frac{\partial u_\theta}{\partial t} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2}\right]$$

$$\frac{\partial u_\theta}{\partial t} = -u_\theta\, \frac{\tau}{(\tau+\tau_1)^2}\, \left(\frac{z^2}{z_a^2} + \frac{r^2}{r_a^2}\right)$$

$$\frac{\partial u_\theta}{\partial r} = u_\theta\, \sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) = \frac{u_\theta}{r}\left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right) + r\left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right)^2 - \frac{\sqrt{\frac{r}{r_i}}(r_a^2 + 12r^2)}{4r_a^2 r^2}\right)\right)$$

$$\frac{\partial^2 u_\theta}{\partial z^2} = u_\theta(4\frac{z^2}{z_a^4} - \frac{2}{z_a^2})$$

$\frac{\partial p}{\partial \theta}$ being independent of θ, we have: $\quad a(\theta,z) = \frac{\partial p}{\partial \theta}\theta + b(z)$

According to the precedent calculations:

$$a(\theta,z) = ru_\theta\theta\, \frac{\tau}{(\tau+\tau_1)^2}\, \left(\frac{z^2}{z_a^2} + \frac{r^2}{r_a^2}\right) + \frac{\nu u_\theta\theta}{r}\left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right) + r\left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right)^2 - \frac{\sqrt{\frac{r}{r_i}}(r_a^2 + 12r^2)}{4r_a^2 r^2}\right)\right)$$
$$+\,\nu u_\theta\theta(4\frac{z^2}{z_a^4} - \frac{2}{z_a^2} - \frac{1}{r^2}) + b(z).$$

$$
\begin{aligned}
\frac{\partial p}{\partial z} &= -2z\sqrt{\frac{\pi}{2}}\ \frac{r_a u_\theta^{o^2}}{z_a^2 r_i}\ e^{-\frac{2z^2}{z_a^2}}\ \mathrm{erf}(\sqrt{2}\frac{r}{r_a}) + \frac{\partial a(\theta,z)}{\partial z} \\
&= -2z\sqrt{\frac{\pi}{2}}\ \frac{r_a u_\theta^{o^2}}{z_a^2 r_i}\ e^{-\frac{2z^2}{z_a^2}}\ \mathrm{erf}(\sqrt{2}\frac{r}{r_a}) - 2\frac{z}{z_a^2} r\theta u_\theta \frac{\tau}{(\tau+\tau_1)^2}\ \left(\frac{z^2}{z_a^2} + \frac{r^2}{r_a^2}\right) + r\theta u_\theta \frac{\tau}{(\tau+\tau_1)^2}\ \left(\frac{2z}{z_a^2}\right) \\
&\quad - 2\frac{z}{z_a^2}\ \frac{\nu u_\theta \theta}{r}\ \left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right) + r\left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right)^2 - \frac{\sqrt{\frac{r}{r_i}}(r_a^2 + 12r^2)}{4r_a^2 r^2} \right)\right) \\
&\quad + \nu\theta u_\theta \left(-8\frac{z^3}{z_a^6} + \frac{12z}{z_a^4} + \frac{2z}{z_a^2 r^2}\right) + \frac{\partial b(z)}{\partial z} \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
b(z) &= \nu u_\theta \theta \left(\left(-4\left(\frac{z_a^2 + z^2}{z_a^4}\right) + 6\frac{1}{z_a^2} + \frac{1}{r^2}\right) - \frac{1}{r}\ \left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right) + r\left(\sqrt{\frac{r}{r_i}}\left(\frac{1}{2r} - \frac{2r}{r_a^2}\right)^2 - \frac{\sqrt{\frac{r}{r_i}}(r_a^2 + 12r^2)}{4r_a^2 r^2} \right)\right)\right) \\
&\quad + r\theta u_\theta \frac{\tau}{(\tau+\tau_1)^2} - r\theta u_\theta \frac{\tau}{(\tau+\tau_1)^2}\ \left(\left(\frac{r^2}{r_a^2} + 1\right) + \frac{z^2}{z_a^2}\right) - \sqrt{\frac{\pi}{8}}\frac{r_a}{r_i} u_\theta^{o^2} e^{\frac{-2z^2}{z_a^2}}\ \mathrm{erf}(\sqrt{2}\frac{r}{r_a}) + \mathrm{A}.
\end{aligned}
$$

$$
P(r,\theta,z,t) = \mathrm{A}.
$$

This pressure expression is a smooth $C^\infty(\mathbb{R}^3 \times [0,\infty])$ function. The velocity u being made of $C^\infty(\mathbb{R}^3 \times [0,\infty])$ smooth functions linked with usual mathematical operations, it is also a smooth $C^\infty(\mathbb{R}^3 \times [0,\infty])$ function.

It has been verified that the presented fluid model satisfies all the conditions to be a three dimensions Navier-Stokes equations solution proposition.

References

Wikipedia and the Clay Math Institute problem definition.

I understand the fact that what follows can be considered as very strange and very new, but it is just a self intention that I give.

In case of recognition of my work as exact, believing sincerely you understand I want to keep my tranquility, and the potential effects of such a recognition in my life, the ability to walk in the streets quietly, and to travel freely wherever I want without being disturbed, beyond other basic freedom rights, I ask you to respect the fact that I want to remain anonymous and not to search who I am, and ask any reader to understand this will. I also want to ensure anyone that I naturally want to continue to contribute to the human knowledge fields, as I may.

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