NATURE'S STARTLING CLUE --

CONFORMAL SYMMETRY

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Abstract: Given the near universality of the Benford/Newcomb Law of First Digits, and the ubiquity of fractal self-similarity observed throughout nature from the smallest to largest scales, it is argued that nature's most fundamental geometry may be conformal geometry, which is universally present in full, partial, and discrete/broken forms. As a consequence, the assumption of *absolute scale* may apply only in restricted contexts, whereas *relative scale* may be the more dominant principle for the cosmos as a whole.

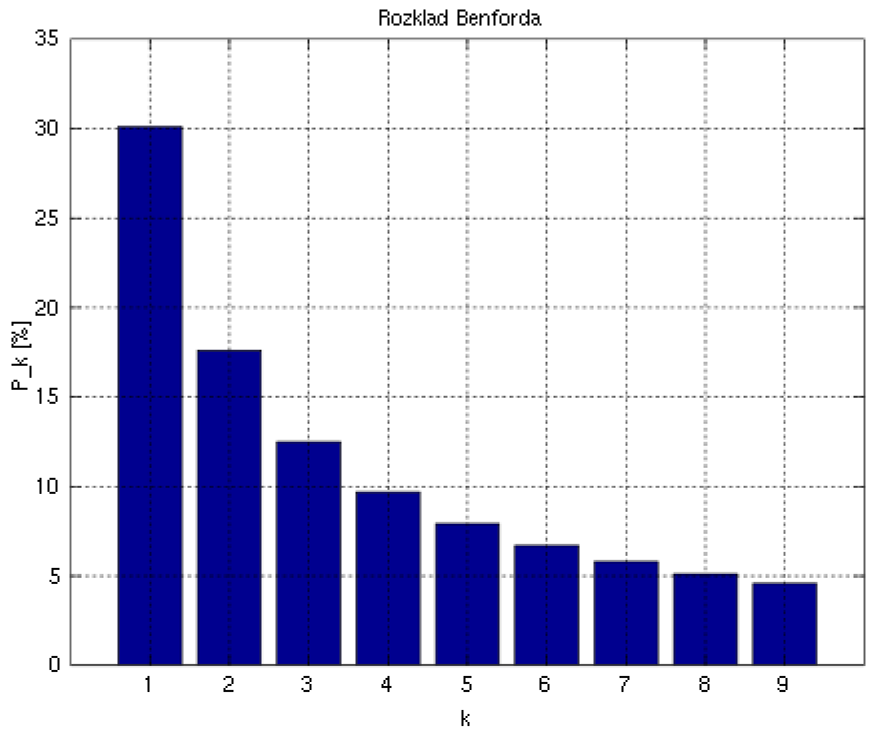
I. Simon Newcomb

In the 1880s Simon Newcomb, a noted astronomer and mathematician, discovered something very odd. It was well before fancy calculators and so when scientists had to do complex mathematical calculations, they needed to use logarithm tables. Newcomb noticed that the first couple of pages (associated with numbers beginning with 1 and 2) were quite dog-eared, but the last pages (associated with numbers beginning with 8 and 9) were far less worn. This was a great mystery to Newcomb since he and apparently all those before him assumed that the first digits of any sizeable collection of data would fall on the numbers 1 through 9 with equal probability (i.e., about 11.11% for each of those first digits). If that were true, then the logarithm table pages should be equally dog-eared. Clearly they were not! How could one explain such a bizarre phenomena.

After considerable thought, and perhaps some checking of various data tables, Newcomb came up with the following hypothesis. The numbers *were* evenly spaced *if* one chose to plot them on a logarithmic scale, not on the usual arithmetic scale. On a log scale the x-axis length devoted to numbers beginning with 1 is about six times wider than the x-axis length devoted to numbers beginning with 9. This would explain the mystery of the dog-eared logarithm tables, and Newcomb published a short paper in the American Mathematical Monthly[1] (where he was an editor) in which he concluded: “The law of probability of the occurrence of numbers is such that all mantissae [first significant digits] of their logarithms are equally probable.” This was a heuristic probability law. Hill notes[2] that Newcomb “supplied neither a precise domain or meaning to this probability, a formal argument, nor numerical data.” Mathematically,

$$\text{Prob } (1^{st} \text{ signif. digit d}) = \log_{10} (1 + 1/d), \qquad d = 1, 2, 3, \ldots 9.$$

Numerically, this probability sequence turns out to be: 30.1%, 17.6%, 12.5%, 9.7%, 7.9%, 6.7%, 5.8%, 5.1%, 5.1% and 4.6%. In graphic form[3], it looks like this:



Unfortunately, no one knew what to make of Newcomb's strange law and it was for the most part forgotten – until the 1930s.

II. Frank Benford

Frank Benford worked at General Electric as a physicist in the 1930s, and he apparently independently discovered the same oddity in logarithm tables that Newcomb had found. He was fascinated with this discovery and devoted years to researching it empirically. He researched roughly 20 data tables, including about 20,000 entries, and the data tables were from many different sets of data from a very diverse selection of physical phenomena. Testing the data for the log-normal first digit distributions, he rediscovered what Newcomb had intuited. In 1938 he published his results[4] and this time many people were impressed with Benford's discovery, and curiously baffled by the fact that the Benford/Newcomb Law of First Digits is so common in nature, math and social phenomena. Benford's somewhat philosophical interpretation of the First Digit Law was than mankind counts arithmetically (1, 2, 3, 4, …) but Nature counts geometrically (e^0, e^x, e^{2x}, e^{3x}, …) "and builds and functions accordingly"[5]. However, this seems to beg the question of why nature should do this, i.e., what is the causal explanation for the heuristic law. How do log-normal distributions come to be ubiquitous in the physical, biological, mathematical and social realms?

Since Benford's paper was written many examples of data sets that obey the First Digit Law have been identified. A *partial* list can include the following.

surface areas of rivers
molecular weights
sizes of stored computer files
atomic element/isotope masses
E1 atomic transition lines in plasmas
pulsar physical properties
universal physical constants
populations of 3,000 countries
surface areas of countries
full widths (lifetimes) of mesons and baryons
Dow Jones numbers
fibonacci sequence
half-lives of radioactive nuclei
internet connections
exoplanet masses, radii, volumes, orbital periods,...
distances to galaxies
distances to stars in our galaxy
death rates
blackbody radiation
prime numbers
river lengths
size of bank accounts

Clearly there must be a scientific explanation for this remarkable ubiquity of log scale distributions, and there have been many attempts. As yet, however, no single explanation has garnered wide acceptance. The hunt for the meaning of this nearly universal phenomena is still ongoing.

III. Toward An Understanding Of The Benford/Newcomb Law of First Digits

Since Benford's paper was published there have been several interpretations of the Benford/Newcomb Law, ranging from philosophical claims of a universal harmony to skeptical proposals that it was an artifact of human uses of numbers, base systems, the floating decimal system, etc. Benford argued that the first digit law might be explained if the data tables were comprised of data garnered from geometric sequences and Prof. Roger Pinkham of Rutgers University argued that some form of relativity theory was probably involved[5]. There is now quite a large body of published work on the Benford/Newcomb Law, but the research papers of the mathematicians Riami[5,6] and Hill[2,7,8] offer the clearest and most convincing discussions of this complicated subject.

1. *Geometric sequences* (2, 4, 8, 16, 32, 64, 128, …) will tend to conform to the logarithm law if they are continued out long enough.

2. *Scale invariance*, wherein physical objects/systems, laws, and processes are largely unaffected by changes in the scales of length, energy and other variables (being multiplied by conserved constants) also tend to conform to the logarithmic law. This also means that if a data table obeys the Benford/Newcomb Law in one set of units, it will obey the logarithm law for any arbitrary choice of units.

3. *Linear recursion*, wherein the same operation is applied in a series of iterations will also tend to conform to the logarithm law. Linear regression tends to generate sequences that are geometric sequences when carried out long enough.

4. The Benford/Newcomb Law is virtually *base invariant* (i.e., base 10, base 2, base e, …).

5. Hill[8] has published "a formal rigorous proof that *the log law is the only probability distribution which is scale invariant, and the only one which is base invariant* (excluding the constant 1)." [emphasis add]

So it might seem like we are have some solid evidence for explaining why the Benford/Newcomb Law is so common in nature and human endeavors. However, there are some caveats that we must consider.

1. First and foremost is the fact that the Benford/Newcomb Law of First Digits is *not* an exact mathematical law. Rather it is a law about distributions that are *unbiased* and cover a large range of scale. There are data tables that do not conform to the law, such as tables of random numbers, or specific heat measurements because they occur over an extremely limited scale range, or the digits of pi.

2. One can improve the agreement with the Benford/Newcomb Law by increasing the size of one data set, or by combining multiple unbiased data sets and the taking a random sample from the union of the different data sets.

One is left with the fact that the Benford/Newcomb Law is a heuristic/empirical law of probability distributions that are amazingly common in the physical world and in human affairs. Moreover, even after nearly 140 years and a large amount of mental effort by mathematicians, physicists and natural philosophers, we still do not have a convincing answer for why nature is this way. As Raimi wrote in his popular Scientific American essay,[6] "Thus all the explanations given so far seem to lack something of finality … the answer remains obscure."

On the other hand, one cannot help but notice that the roles played by geometric sequences, scale invariance, and linear regressions have a common thread running through them that might be stated as repeating the same structure, law, operation or process on many different scales, or more simply stated: *same thing on different scales*. With this simple principle one can understand how each of the data sets listed above (e.g., river lengths, bank account balances, the Fibonacci sequence, atomic masses, and exoplanet orbital periods) conforms to a log law probability distribution for first digits. Before proceeding to a possible answer to the meaning of the Benford/Newcomb Law, we need to consider another phenomenon that is *ubiquitous* in the natural world and human affairs: *self-similarity*.

IV. Mandelbrot

In the second half of the 1970s Benoit Mandelbrot introduced the public and the scientific community to fractals and fractal geometry. He demonstrated that mathematical and physical objects could have fractional dimensionality, rather than the integer dimensionality that had always ruled geometry in the past. Mandelbrot gathered diverse data and information showing that fractal self-similarity was evident in our everyday world, as well as in the microcosm and cosmocosm. Fractal geometry and fractal modeling have since played important roles in all the physical sciences (e.g., astronomy, physics, meteorology, and geology), in the life science (e.g., biology, genetics, medicine and neurobiology), in the social sciences (e.g., sociology, city planning, economics), in the technical sciences (e.g., materials science, computer science and logistics), and in the mathematical sciences. Every year 100s of papers containing the word *fractal* in their abstracts are submitted to the arxiv.org preprint servers, and that only counts papers in the physical sciences!

Here is a very short list of a few of the well-known examples of fractal structures and processes in the observable world.

Filamentary cosmic web on the largest observable scale
Branching patterns of trees
Veins of leaves
Cloud morphology
Circulatory systems in organisms
Neuron organization and connectivity in brains
Galactic clustering (galaxies, galactic groups, clusters of groups, clusters of clusters, …)
Architecture of lungs (15 levels of self-similar tube branching)
Coastlines and geographic boundaries everywhere
Base-pairing and long-range correlations in DNA
Open-shut temporal patterns of sodium, calcium and potassium channels in cells
Clustering of stars within galaxies
Clustering of atomic ions in the high-energy plasmas comprising stars
Music of Bach and many others have self-similar motifs
Morphology of Interstellar gas/dust clouds
Temporal fluctuations of the solar wind

Multiple analogies between atomic nuclei, stellar pulsars and galactic quasars
Atmospheric turbulence of many types
Snowflake morphologies
Microscopic Brownian motion of very many physical systems
Waves on ocean surfaces
Wall art of medieval churches and mosques
Fluid turbulence with all bodies of water
Mixing of different fluids
Tributary and delta drainage systems of rivers
Peak/valley mountain range morphology
Sizes and distributions of astronomical cratering
Very many landscapes viewed from the air: desert, arctic, forest, lake regions, …
Ferns, cedar boughs, Dill plants, cactus plants, broccoli, and many more plant types
Organization of governments, judicial systems, law enforcement agencies, etc.
The internet's node structuring, networking and temporal operation, etc.
Growth patterns and interconnectedness of cities
Mathematics: proofs of the Pythagorean theorem, the Golden section, M-set, …

One of the defining features of fractals is their *self-similarity*. When something has the property of self-similarity, it means that there are multiple copies of the same (or similar) shapes, physical systems, operational processes or temporal phenomena on different spatial and/or temporal scale. The self-similarity can be merely statistical, or moderately strong, or even exact. One can find examples of *continuous* self-similarity (uniform over applicable scale range), or the more common *discrete* self-similarity that manifests itself at a discrete set of scales. Here again we see a truly ubiquitous feature of our world and once again the fundamental underlying principle can be briefly summarized as: **same thing on different scales**.

Since I firmly believe in Einstein's well-known dictum: "Raffiniert ist der Herrgott, aber boshaft ist er nicht" (i.e., "Subtle is the Lord, but malicious He is not"), I am highly inclined to think that nature is trying to tell us something important that we have missed, and that the "good Lord" (which was Einstein's affectionate name for Spinoza's god = nature) would not mislead us in such a dramatic way. So maybe we are ready to attempt a decoding of the message: *same thing on different scales*.

V. Toward An Explanation Of The Benford/Newcomb Law and Fractal Self-Similarity

We already have stated several times that the common thread running through the scale invariance, linear recursion, geometric sequences and self-similarity that underlie the logarithmic law of first digits and fractal self-similarity is that structures, physical laws, mathematical operations and various processes are repeated with little variation on different scales. This seems to suggest that, whenever possible, nature treats scale in a *relational or relative* way, rather than in an absolute way.

What is surprising is that most of our major theories in the physical sciences, such as the standard model of particle physics, quantum mechanics, general relativity and the standard model of cosmology, being based mainly on Euclidean geometry and non-Euclidean geometry, all involve, for the most part, *absolute global scale* when measurements are made in a proper rest frame. For example, the hydrogen atom is thought to have a fixed Bohr radius for its ground state and a fixed proper mass. **So it appears that we have a conflict here. On one hand we have nature (and to a restricted sense relativity theory) advocating for relative scale, while on the other hand we have the majority of the dominant physics theories of the 21st century firmly founded on the assumption of absolute scale.** One should also note here that the equally firm belief among physicists in strict (or at least strong) *reductionism* is inextricably intertwined with the assumption of absolute scale.

One wonders if there might be a way to resolve this fundamental conflict so that our observations of nature and our theoretical models of how nature works are in less disagreement with each other. Happily, I think there most certainly is and it is called conformal geometry, which has the potential to lead us into the less restrictive realm of relative scale without losing the beauty and knowledge-generating power of our best physics theories, although it would require recasting them so that they are compatible with the symmetries of conformal geometry.

Briefly, conformal geometry preserves shapes and angles, but has no fixed lengths, and so it is fully compatible with relative scale. The conformal symmetry group contains the translations, rotations and relativistic boosts that we are most familiar with in the reigning 10-parameter Poincare group, but adds another 5 symmetries: 1 dilatation and 4 special conformal transformations relating to combinations of rotations and translations (but these are beyond the scope of this discussion). Maxwell's theory of electromagnetism has all the conformal symmetries when limited to massless electric and magnetic fields. The massless “vacuum” equations of General Relativity also possess full conformal symmetry. However, adding charges and masses to these theories is believed to “break” their conformal symmetry, although attempts have been made to circumvent these problems. Conformal geometry and conformal field theories have been applied in a somewhat restricted manner in quantum electrodynamics, general relativity (with limited success), and various theories of particle physics.

It is the *dilatation symmetry* (often simply called *dilation* symmetry) that is of most interest to us here. Dilation invariance means that a shape, or the physical properties of a system, or relevant laws and processes remain the same for different changes of scale. *So here we once again meet our theme: same thing on different scales*. Given that self-similarity (copies within copies within copies,…) can be understood in terms of a *discrete subset* of structures or processes taken from full and continuous conformal symmetry, one might think that fractal geometry may be synonymous with conformal geometry in full, partial or discrete/ broken form, depending on the specific limitations of each case.

Past theoretical applications of scale invariance, self-similarity and conformal symmetries have usually been restricted to finite ranges of scale. The examples of the data tables that conform to the logarithmic law and the well-recognized examples self-similarity observed in nature cover a vast range of *total* combined scale, but individually the examples are usually limited to either the microscopic, macroscopic or cosmic ranges of scale. This is probably due to the fact that the world is a very complicated place with a large number of distinct structures, laws and processes that must compete interactively, and this is likely to interfere with full conformal symmetry. So perhaps we must be satisfied with partial and broken forms of conformal symmetry. That is what the Benford/Newcomb Law of First Digits and the ubiquitous fractal self-similarity in our observable world indicate so far. And yet, one wonders if nature might have one more spectacular surprise in store for us.

VI. “One Of The Most Exquisite Conjectures In Science Or Religion”

Speaking of the general concept that nature might repeat itself in a conformally symmetric manner on radically different size scales, Carl Sagan offered the following comment in his book, Cosmos[9], which was linked to the TV series of the same name.

"There is an idea -- strange, haunting, evocative – one of the most exquisite conjectures in science or religion. It is entirely undemonstrated; it may never be proved. But it stirs the blood. There is, we are told, an infinite hierarchy of universes, so that an elementary particle, such as an electron, in our universe would, if penetrated reveal itself to be an entire closed universe. Within it , organized into the local equivalent of galaxies and smaller structures, are an immense number of other, much tinier elementary particles, which are themselves universes at the next level, and so on forever – an infinite downward regression, universes within universes, endlessly. And upward as well."

Sagan's vision of an infinite self-similar cosmos resonated with me because several years earlier I had experienced a similar epiphany. It was an epiphany that had also occurred to Democritus in the 5th century BC, to Spinoza in the 17th century, to Kant in the 18th century, and to many others over the last 2500 years. In his last writing, for a 1955 conference in Italy celebrating of 50 years of relativity, Einstein[10] noted the potential for solutions of his latest unified field equations that were "similar but not congruent", i.e., that exhibited discrete self-similarity and dilation invariance. He was skeptical, however, because he noted that atoms appear to have fixed radii and masses, and this seemed to conflict with the concept of relative scale. Alas, he died about a month later that same year and never had a chance to fully explore the hint he found in those equations. Had he lived longer, Einstein might have considered Sagan's "exquisite conjecture" of global discrete dilation symmetry, i.e., of a cosmos that repeats itself, but only in the case of almost unimaginably large and discrete (i.e., quantized) jumps of scale.

In my case, while studying at the University of Washington in Seattle during the early 1970s, the *hierarchical organization of nature* had always seemed self-evident to me. It also seemed obvious that the cosmological hierarchy was *highly stratified* into atomic, stellar and galactic scales. Whereas the observable portion of the cosmic hierarchy had a mass range of over 80 orders of magnitude (!), three relatively narrow mass ranges of about 5 orders of magnitude for each strongly dominated the hierarchy: the *Atomic Scale* (particles, ions and atoms), the *Stellar*

Scale (black holes, neutron stars and main sequence stars), and the *Galactic Scale* (globular clusters, quasars and galaxies). These fundamental cosmological *Scales* together represented less than 1/5 of the full mass range, and yet they contained nearly all of the mass of the observable universe.

I also began to notice that there were many possibilities for drawing *analogies* among the systems on the different *Scales*. For example, one can see quite a few similarities among atomic nuclei, pulsars, and quasars. Then on the night of December 21st in 1976, I was sitting in the reading room of the Marine Biological Laboratory in Woods Hole, Massachusetts and perusing a *Nature* paper[11] by E. R. Harrison entitled "Electrified Black holes". Harrison noted that stars would have a net positive charge because their high temperatures would preferentially drive off the lighter electrons and retain heavier nuclei which are positive. The concept of positively charged stars, combined with a wealth of disparate clues for stellar-atomic analogies, suddenly caused me to envision the same strange new paradigm that had 'stirred the blood' of Sagan: that of an infinite and eternal hierarchical cosmos in which the same laws, structures, kinematics, and dynamics were repeated endlessly on vastly separated *Scales*. Even though I was fully conscious of how crazy the idea sounded, and how exceedingly speculative it was at that point, a life-long journey to explore this radical idea had begun in earnest.

The first thing one needed to test the idea against observational knowledge was to derive a set of length (L), time (T) and mass (M) transformation equations that relate the physical properties of the physical systems on the atomic, stellar, and galactic *Scales*. After inventorying all known physical objects from electrons to superclusters of galaxies, and studying their kinematics and dynamics, I was able to choose the most reasonable sets of analogue pairs from differing Scales, and use them to establish the following transformation equations for neighboring *Scales* n and n+1.

$$L_n = \Lambda L_{n+1}$$

$$T_n = \Lambda T_{n+1}$$

$$M_n = \Lambda^D M_{n+1}$$

Lambda (Λ) is equal to 5.2×10^{17}. The mass scaling requires the fractional exponent D = 3.174, which is appropriate for a fractal theory, and for 3-dimensional structures in our 4-dimensional world. These amazingly simple *Scale* transformation equations have never required adjustments (i.e., fudging) since they were established decades ago. They are highly and specifically consistent with discrete scale invariance, self-similarity and discrete dilation symmetry; they also generate geometric sequences and can be viewed as examples of linear recursion. With these simple equations that embody our "same thing on radically different *Scales*" principle, one can predict and retrodict quite a few properties of the observable universe. A *partial* list would include:

Successful prediction of pulsar-planets before their discovery
Radius of the proton
Mass of the proton
Abundance of red dwarf stars
Abundance of white dwarf stars
Lower limit radii for red dwarf stars
Average radii for white dwarf stars
Range of radii for white dwarf stars
Overall range of radii for stars in general
Range of radii for galaxies
Typical spin periods of pulsars
Typical spin periods of galaxies
Geometrical shapes of atomic nuclei and galaxies
Average mass of white dwarf stars
Lower mass limit of white dwarf stars
K_s/K_a ratio for the $J = K_nM^2$ relationships of stars and atoms
Δ_s/Δ_a ratio for the $\mu = \Delta_iJ$ relationships on the Stellar and Atomic Scales
Global 160 minute g-mode oscillation of the Sun
Magnetic dipole moment ranges for atomic nuclei and neutron stars
Preferred periods for white dwarf stars
Range of oscillation periods for neutron stars
Keplerian period-radius laws for variable stars and Rydberg atoms
Ratio of the K_s/K_a from the $P^2 = K_iR^3$ relationships of atoms and stars
Dark matter mass peaks at 8×10^{-5}, 0.15 and 0.58 solar mass
Full dark matter mass range is 8×10^{-5} to 35 solar mass
Steep drop in stellar mass function below 0.15 solar mass
Gap in the white dwarf mass function at 0.73 solar mass
Prediction of unusually low exoplanet abundance for lowest mass red dwarf stars
Preferred mass peaks in the white dwarf mass spectrum
Decreased upper limit for the masses of single stars
Revised upper limits to the observed radii of stars and atoms
Active galaxy oscillation periods on the order of 10^7 years

Self-similar scaling between stellar activity cycles and e^- spin transitions
Self-similarity between RR Lyrae stars and specific E-state transitioning of He atoms
Self-similarity between δ Scuti stars and excited C atoms
Self-similarity between ZZ Ceti stars and excited He^+ ions
Self-similarity between SX Pheonnicis stars and excited boron atoms
Approximate radius of the alpha particle
Potential resolution of the vacuum energy density crisis
Successful prediction of billions of unbound planetary-mass objects

If one were interested in learning more about what I refer to as Discrete Scale Relativity, then there is a website[12] [http://www3.amherst.edu/~rloldershaw] which includes a main ideas section, a selection of the most important published papers, a full listing of 70 publications, and a wealth of new developments and discoveries that are way beyond the scope of this paper. A review of 15 definitive predictions, including brief discussions of motivating evidence and accumulated observational support, can be obtained at https://www.academia.edu/2917630/Predictions_of_Discrete_Scale_Relativity .

At any rate, it is definitely possible that nature has at least one more amazing surprise in store for us, when we are ready for it. Based on the ubiquity of the Benford/Newcomb Law of First Digits, the ubiquity of fractal self-similarity, and the evidence for Sagan's exquisite conjecture of nature repeating itself on different cosmological *Scales*, one can justifiably claim that absolute scale is in serious doubt, that anything more than limited reductionism within *Scales* is also in serious doubt, and that global discrete conformal symmetry needs to be aggressively explored. The emerging relative scale paradigm would require major changes in some of the most basic and firmly held assumptions of current theoretical physics. Such changes will not come quickly or without strong resistance. In fact, this radical paradigm change may have to await a new generation of physics students and natural philosophers who are less indoctrinated when it comes to long-held assumptions. Perhaps they will discover that nature is infinitely more subtle, elegant and unified than most of their predecessors have dared to dream.

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