Abstract

I provide for a coherent theory of vacuum quantum gravity.

A formulation of vacuum quantum relativity.

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1 Introduction

A prototype theory of quantum gravity is developed based upon my previous work on covariant quantum mechanics. The old problem of the diffeomorphism and Hamiltonian constraints are solved in a fairly unique way providing for a complete background independent propagator.

2 Quantum gravity; old fashioned metric formulation.

Let $\mathcal{M}$ be a $n+1$ dimensional manifold with initial and final boundaries Σ_1 and Σ_2 which we shallassume to be spacelike regarding any further metric specification. Consider a metric $g_{\alpha'\beta'}(x,y)$ where the primed indices refer to the y coordinate and unprimed ones to the x coordinate. Actually, g is a bitensor on the product manifold $\mathcal{M} \times \mathcal{M}$ with the ordinary product differentiable structure at least what concerns the second factor; mathematically, it belongs to $\mathcal{M} \times T_2\mathcal{M}_{\mathbf{sym}}$. The data on Σ_i are specified by a vierbein E_a and a vacuum solution $g_{\alpha'\beta'}(x,y)$ for any $x \in \Sigma_1$ and $y \in \mathcal{M}$ with initial data $h_{\alpha'\beta'}(x,y) = h_{\alpha'\beta'}(y) = \sum_i E_{i\alpha'}(y)E_{i\beta'}(y)$ where $E_i \in T^1\Sigma$ satisfying the diffeomorphism constraints $Z_i(h)(y) = 0$. Now, every $h \in T^2\Sigma$ defines a Fourier transform in $L^2(\Sigma, h)$:

$$\pi_h(y) = \int_{\mathbb{R}^n} d^n k \widehat{\pi_{x,h}}(k)\psi_{\Sigma,h}(x,y,k)$$

where $\psi_{\sigma,h}(x,y,k)$ has been defined in my previous book on covariant quantum mechanics. We shall henceforth assume this Fourier transform to define a diffeomorphism modulo a certain ambiguity on the function space $L^2(\mathbb{R}^n, d^n k, x, h)$ which is modelled by linear operators $\Psi_{x,h} : L^2(\mathbb{R}^n, d^n k, x, h) \to L^2(\mathbb{R}^n, d^n k, x, h)$ which are not necessary unitary with respect to the standard inner product $\langle \pi|\psi\rangle_{x,h} = \int d^n k \widehat{\pi_{x,h}}(k)\widehat{\psi_{x,h}}(k)$ where some representants $\widehat{\psi_{x,h}}, \widehat{\pi_{x,h}}$ have been chosen. Notice that these expressions are not necessarily the same as $\int d^n y \pi(y)\psi(y)$ and are moreover heavily x dependent. Choose any minimal subspace $\mathcal{Z}_{x,h}$ of $L^2(\mathbb{R}^n, d^n k, x, h)$ containing one representant only and which glue nicely together to a Hilbert bundle

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on Σ modelled on $T^1\Sigma$. Likewise, it is possible to consider the space of initial data $\pi_{\alpha'\beta'}(y)$ on $T^2\Sigma$ and define a Fourier transform defined by

$$\widehat{\pi_{ab;x,h}}(k)$$

where a, b are Euclidean indices in x and the latter is a function in $\mathcal{Z}_{x,h}$. For more details, see a previous publication on the Fourier transform. Point is that the Hamiltonian constraint $H(h, \pi) = 0$ together with the diffeomorphism constraints must be rewritten in $\otimes_{\frac{n(n+1)}{2}} \mathcal{Z}_{x,h}$ which defines an infinite dimensional Riemannian submanifold $\mathcal{C}_{h,x}(\pi)$ in the induced metric. A canonical limiting measure may now be defined on this infinite dimensional Riemannian manifold by putting k space in finite balls of radius L and taking periodic boundary conditions in the radius as well as modes with integer factor m bounded by $|m| \leq N$, so that the measure spaces $\mathcal{Z}_{x,h}(L, N)$ are finite dimensional. The natural thermodynamic limit is $N \to \infty$ followed by $L \to \infty$ in every $x \in \Sigma$. A natural inner product space and measure are therefore defined by means of these truncations and a scalar product between different (h, π) data in $\otimes_{n(n+1)} \mathcal{Z}_{x,h}$ is given by the linear extension in the π factor of

$$\langle (h_1, \pi_1)|(h_2, \pi_2)\rangle = \int_\Sigma d^n x (h_1(x) h_2(x))^{\frac{1}{4}} \int d^n k \widehat{\pi^1_{x,h_1}}(k) \widehat{\pi^2_{x,h_2}}(k)$$

and likewise so for the Riemannian metrics themselves. This scalar product has no suitable additive properties with respect to the spatial metric. The respective scalar product has a differentiable action of Diff(Σ) by means of unitary operators and therefore the identity component of the spatial diffeomorphism group is generated by unitary operators.

This brings us back to the idea that a spacetime in the state $|h\rangle$ represents the birth of an entire universe and that the appropriate integration over different metrics is done by means of this "background" metric. This intertwining between dynamics and kinematics was present already in my covariant quantum theory published herefore and it canonically defines spacetime particles and the appropriate propagators associated to the birth of a spatial universe in contrast to the standard quantum theory on Minkoswki where the maximal symmetry makes this association canonical. Specifically, denote by the metric h on the boundary Σ the birth metric of a spatial universe; then a wave function Φ in h' satisfying $Z_i(h') = 0$ can be fully developed with respect to h. That is,

$$h'_{\alpha'\beta'} = \Gamma^a_{\alpha'}(x, y)\Gamma^b_{\beta'}(x, y) \int d^n k \widehat{h'}_{ab;x,h}(k)\psi(x, y, k, x, h)$$

assuming that x, y are connected by exactly one h geodesic and $\Gamma^a_{\alpha'}(x, y)$ is the propagator along that geodesic. A more complicated but equally adequate formula exists when multiple h geodesics connect x with y. The diffeomorphism constraints on h' are most easily expressed in terms of integral equations of the Fourier components $\widehat{h'}_{ab;x,h} \in \mathcal{Z}_{x,h}$. Hence, $\Phi(h')$ must be decomposed with respect to the induced metric on the h' submanifold $\mathcal{D}(h, x)$ in $\otimes_{\frac{n(n+1)}{2}} \mathcal{Z}_{x,h}$ defined by the constraints. The usual L, N filtration gives then a sequence of measures $d\mu_{L,N}$ and Fourier transforms defined by the canonical flat metric on tangent space such that

$$\Phi_{h,x}(h') = \lim_{L,N\to\infty} \int_{T_{h,x}\mathcal{D}(h,x)(L,N)(h)} d\pi_{ij;x,h} \widehat{\Phi_{x,h;L,N}}(\pi_{ij;x,h})\psi(h, h', x, \pi_{ij;x,h})$$

where $\pi_{ij;x,h}$ are the cotangent bundle coordinates in a orthogonal basis of $T_{h,x}\mathcal{D}(h,x)(L,N)(h)$ and ψ denotes the usual Fourier functions. The Fourier decompositions differ when varying the point x but crucially depend upon h and one canonically defines a canonicaldiffeomorphism invariant scalar product

$$\langle \Psi_h | \Phi_h \rangle = \lim_{L,N \to \infty} \int_\sigma d^n x \sqrt{h(x)} \int_{T_{h,x}\mathcal{D}(h,x)(L,N)(h)} d\pi_{ij;x,h} \widehat{\Psi_{x,h;L,N}}(\pi_{ij;x,h}) \widehat{\Phi_{x,h;L,N}}(\pi_{ij;x,h})$$

for wavefunctions regarding a universe born at (Σ, h). Given (h,π) one solves for the vacuum Einstein equations of motion

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

with any acceptable lapse and shift vector so that all variables remain well defined; for example, the Gaussian gauge could break down before Σ_2 is reached. Now, we wish to study the evolution of $\Psi_h(h')$ in "time". Given a point y in $(\mathcal{M}, g(h,\pi,N_\mu))$ to the future of Σ_1, denote by $P(y,g)$ the subset of pairs $(x,v) \in \Sigma \times T\mathcal{M}_{|\Sigma}$ such that $\exp_x(v) = y$. The spacetime Gaussian $\psi(h,\pi,N_\mu;h',y)$ is given as

$$\psi(h,\pi,N_\mu;h',y) = \int_{P(y,g(h,\pi,N_\mu))} d^n x \sqrt{\tilde{h}(x)} \psi(h,\pi,N_\mu;h',x,v)$$

where $\tilde{h}$ is the metric induced on $P(y,g)$ by means of the product metric $h \otimes \delta$ on $\Sigma \times T_1\Sigma$. The functions $\psi(h,\pi,N_\mu;h',x,v)$ are determined by means of the standard Schroedinger equation

$$\frac{d}{ds}\psi(h,\pi,N_\mu;h',x,v;s) = i\pi_{\mu\nu;x}(\gamma_{x,v}(s)) g^{\mu\kappa}(\gamma_{x,v}(s)) g^{\nu\lambda}(\gamma_{x,v}(s)) h'_{\kappa\lambda;x}(\gamma_{x,v}(s)) \psi(h,\pi,N_\mu;h',x,v;s)$$

with conditions $\gamma_{x,v}(0) = x, \gamma_{x,v}(0) = y$ and the tensors π, h on Σ have canonical extensions towards $\mathcal{M}$. The tensors $\pi_{\mu\nu;x}(\gamma_{x,v}(s))$ are covariantly constant along the g-geodesic and their value at the endpoint may vary from starting point to starting point x. The Hamiltonian constraint is as such preserved along any geodesic and is entirely foliation independent. Keeping N^μ fixed, we therefore can define a propagator

$$D(h,N_\mu,h';x,y) = \lim_{L,N\to\infty} \int_\sigma d^n z \sqrt{h(z)} \int_{\mathcal{C}_{z,h}(L,N)} d\pi_{ij;z,h,L,N} \frac{1}{\pi_{\alpha\beta}(x) h'^{\alpha\gamma}(x) h'^{\beta\delta}(x) \pi_{\gamma\delta}(x)} \psi(h,\pi,N_\mu;h',x,y)$$

which is nothing but the adequate substitute for the graviton propagator on the background (h,π) in a first quantized theory. Alas, as we shall soon see, this object is not of much use. A second quantized theory would demand a four index bitensor propagator with two indices in every reference point each.

The evolution therefore is

$$\Phi_{h,\Sigma_1}(\overline{h}, N_\mu, \Sigma_2) = \lim_{L,N\to\infty} \int_{\Sigma_2} d^n y \sqrt{\overline{h}(y)} \int_{\Sigma_1} d^n x \sqrt{h(x)}$$

$$\int_{\mathcal{C}_{x,h}(L,N)} d\pi_{ij;x,h,L,N} \int_{K(h,\pi,N_\mu,\Sigma_1,x;\overline{h},y,\Sigma_2)} d\mu_{h,\pi,N_\mu,\Sigma_1,x;\overline{h},y\Sigma_2}(h') \Phi_{h,\Sigma_1}(h',\Sigma_1)$$

$$\frac{1}{\pi_{\alpha\beta}(x) h'^{\alpha\gamma}(x) h'^{\beta\delta}(x) \pi_{\gamma\delta}(x)} \psi(h,\pi,N_\mu;h',x,y)$$

where $K(h, \pi, N_\mu, \Sigma_1, x; \overline{h}, y, \Sigma_2)$ is the Hilbert manifold of Riemannian metrics h' on Σ canonically metricized and equipped with the limiting volume form by means of $T\Sigma_x$ such that the projection of

$$\int_{P(y,g(h,\pi,N_\mu))} d^n z \sqrt{\tilde{h}(z)} h'_{\mu\nu;z}(y)$$

on Σ_2 equals $\overline{h}_{\alpha\beta}(y)$.

3 Final Remarks.

Extensions of the latter theory including second quantized matter appear obvious; further exposition of this paper is material for an extended version.